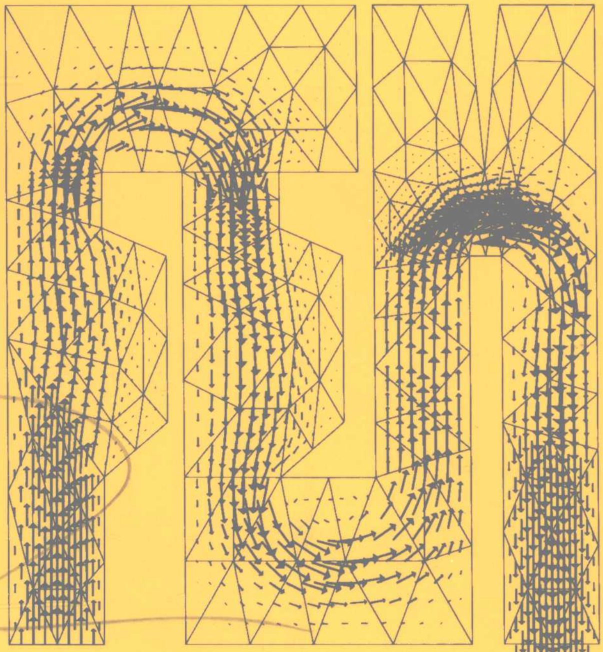
Susanne C. Brenner
L. Ridgway Scott

The Mathematical Theory of Finite Element Methods

Second Edition

有限元方法的数学理论

第2版



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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

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Preface to the Second Edition

This edition contains two new chapters. The first one is on the additive Schwarz theory with applications to multilevel and domain decomposition preconditioners, and the second one is an introduction to *a posteriori* error estimators and adaptivity. We have also included a new section on an example of a one-dimensional adaptive mesh, a new section on the discrete Sobolev inequality and new exercises throughout. The list of references has also been expanded and updated.

We take this opportunity to extend thanks to everyone who provided comments and suggestions about this book over the years, and to the National Science Foundation for support. We also wish to thank Achi Dosanjh and the production staff at Springer-Verlag for their patience and care.

Columbia, SC
Chicago, IL
20/02/2002

Susanne C. Brenner
L. Ridgway Scott

Preface to the First Edition

This book develops the basic mathematical theory of the finite element method, the most widely used technique for engineering design and analysis. One purpose of this book is to formalize basic tools that are commonly used by researchers in the field but never published. It is intended primarily for mathematics graduate students and mathematically sophisticated engineers and scientists.

The book has been the basis for graduate-level courses at The University of Michigan, Penn State University and the University of Houston. The prerequisite is only a course in real variables, and even this has not been necessary for well-prepared engineers and scientists in many cases. The book can be used for a course that provides an introduction to basic functional analysis, approximation theory and numerical analysis, while building upon and applying basic techniques of real variable theory.

Chapters 0 through 5 form the essential material for a course. Chapter 0 provides a microcosm of what is to follow, developed in the one-dimensional case. Chapters 1 through 4 provide the basic theory, and Chapter 5 develops basic applications of this theory. From this point, courses can bifurcate in various directions. Chapter 6 provides an introduction to efficient iterative solvers for the linear systems of finite element equations. While essential from a practical point of view (our reason for placing it in a prominent position), this could be skipped, as it is not essential for further chapters. Similarly, Chapter 7, which derives error estimates in the maximum norm and shows how such estimates can be applied to nonlinear problems, can be skipped as desired.

Chapter 8, however, has an essential role in the following chapters. But one could cover only the first and third sections of this chapter and then go on to Chapter 9 in order to see an example of the more complex systems of differential equations that are the norm in applications. Chapter 10 depends to some extent on Chapter 9, and Chapter 11 is essentially a continuation of Chapter 10. Chapter 12 presents Banach space interpolation techniques with applications to convergence results for finite element methods. This is an independent topic at a somewhat more advanced level.

To be more precise, we describe three possible course paths that can be chosen. In all cases, the first step is to follow Chapters 0 through 5. Someone interested to present some of the "hard estimates" of the subject could then choose from Chapters 6 through 8, and 12. On the other hand, someone interested more in physical applications could select from Sect. 8.1, Sect. 8.3 and Chapters 9 through 11. Someone interested primarily in algorithmic efficiency and code development issues could follow Chapters 6, 8, 10 and 11.

The omissions from the book are so numerous that is hard to begin to list them. We attempt to list the most glaring omissions for which there are excellent books available to provide material.

We avoid time-dependent problems almost completely, partly because of the existence of the book of (Thomée 1984). Our extensive development of different types of elements and the corresponding approximation theory is complementary to Thomée's approach. Similarly, our development of physical applications is limited primarily to linear systems in continuum mechanics. More substantial physical applications can be found in the book by (Johnson 1987).

Very little is said here about adaptivity. This active research area is addressed in various conference proceedings (cf. Babuška, Chandra & Flaherty 1983 and Babuška, Zienkiewicz, Gago & de A. Oliveira 1986).

We emphasize the variety of discretizations (that is, different "elements") that can be used, and we present them (whenever possible) as families depending on a parameter (usually the degree of approximation). Thus, a spirit of "high-order" approximations is developed, although we do not consider increasing the degree of approximation (as is done in the so-called P-method and spectral element method) as the means of obtaining convergence. Rather, we focus on mesh subdivision as the convergence parameter. The recent book by (Szabo & Babuška 1991) may be consulted for alternatives in this direction.

Although we provide a brief introduction to mixed methods, the importance of this subject is not appropriately reflected here. However, the recent book by (Brezzi & Fortin 1991) can be consulted for a thorough treatment of the subject.

We draw extensively on the book of (Ciarlet 1978), both following many of its ideas and using it as a reference for further development of various subjects. This book has recently been updated in (Ciarlet & Lions 1991), which also contains an excellent survey of mixed methods. Moreover, the Handbook series to which the latter reference belongs can be expected to provide valuable reference material in the future.

We take this opportunity to thank the many people who have helped at various stages, and in many different ways, in the preparation of this book. The many students who struggled through early drafts of the book made invaluable contributions. Many readers of the preliminary versions will find their specific suggestions incorporated.

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Cover illustration: Flow of a Newtonian fluid at Reynolds number 58, computed with the techniques developed in Chapters 12 and 13.

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Contents

Chapter 0

Basic Concepts

The finite element method provides a formalism for generating discrete (finite) algorithms for approximating the solutions of differential equations. It should be thought of as a black box into which one puts the differential equation (boundary value problem) and out of which pops an algorithm for approximating the corresponding solutions. Such a task could conceivably be done automatically by a computer, but it necessitates an amount of mathematical skill that today still requires human involvement. The purpose of this book is to help people become adept at working the magic of this black box. The book does *not* focus on how to turn the resulting algorithms into computer codes, although this is at present also a complicated task. The latter is, however, a more well-defined task than the former and thus potentially more amenable to automation.

In this chapter, we present a microcosm of a large fraction of the book, restricted to one-dimensional problems. We leave many loose ends, most of which will be tied up in the theory of Sobolev spaces to be presented in the subsequent chapter. These loose ends should provide motivation and guidance for the study of those spaces.

0.1 Weak Formulation of Boundary Value Problems

Consider the two-point boundary value problem

$$(0.1.1)\qquad \begin{aligned} -\frac{d^2u}{dx^2} &= f \text{ in } (0,1) \\ u(0) = 0, &\qquad u'(1) = 0. \end{aligned}$$

If u is the solution and v is any (sufficiently regular) function such that $v(0) = 0$, then integration by parts yields

$$(0.1.2)\qquad \begin{aligned} (f,v) := \int_0^1 f(x)v(x)dx &= \int_0^1 -u''(x)v(x)dx \\ &= \int_0^1 u'(x)v'(x)dx =: a(u,v). \end{aligned}$$

Let us define (formally, for the moment, since the notion of derivative to be used has not been made precise)

$$V = \{v \in L^2(0,1): \quad a(v,v) < \infty \text{ and } v(0) = 0\}.$$

Then we can say that the solution u to (0.1.1) is characterized by

$$(0.1.3) \qquad u \in V \quad \text{such that} \quad a(u,v) = (f,v) \qquad \forall v \in V,$$

which is called the *variational* or *weak* formulation of (0.1.1).

The relationship (0.1.3) is called "variational" because the function v is allowed to vary arbitrarily. It may seem somewhat unusual at first; later we will see that it has a natural interpretation in the setting of *Hilbert spaces.* (A Hilbert space is a vector space whose topology is defined using an inner-product.) One example of a Hilbert space is $L^2(0,1)$ with inner-product $(\cdot,\cdot)$. Although it is by no means obvious, we will also see that the space V may be viewed as a Hilbert space with inner-product $a(\cdot,\cdot)$, which was defined in (0.1.2).

One critical question we have not yet dealt with is *what sort of derivative is to be used* in the definition of the bilinear form $a(\cdot,\cdot)$. Should this be the classical derivative

$$u'(x) = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \,?$$

Or should the "almost everywhere" definition valid for functions of bounded variation (BV) be used? We leave this point hanging for the moment and hope this sort of question motivates you to study the following chapter on Sobolev spaces. Of course, the central issue is that (0.1.3) still embodies the original problem (0.1.1). The following theorem verifies this under some simplifying assumptions.

(0.1.4) Theorem. *Suppose* $f \in C^0([0,1])$ *and* $u \in C^2([0,1])$ *satisfy* (0.1.3). *Then* u *solves* (0.1.1).

Proof. Let $v \in V \cap C^1([0,1])$. Then integration by parts gives

$$(0.1.5) \qquad (f,v) = a(u,v) = \int_0^1 (-u'')v\,dx + u'(1)v(1).$$

Thus, $(f - (-u''), v) = 0$ for all $v \in V \cap C^1([0,1])$ such that $v(1) = 0$. Let $w = f + u'' \in C^0([0,1])$. If $w \not\equiv 0$, then $w(x)$ is of one sign in some interval $[x_0, x_1] \subset [0,1]$, with $x_0 < x_1$ (continuity). Choose $v(x) = (x-x_0)^2(x-x_1)^2$ in $[x_0,x_1]$ and $v \equiv 0$ outside $[x_0,x_1]$. But then $(w,v) \neq 0$, which is a contradiction. Thus, $-u'' = f$. Now apply (0.1.5) with $v(x) = x$ to find $u'(1) = 0$. Of course, $u \in V$ implies $u(0) = 0$, so u solves (0.1.1). □

(0.1.6) *Remark.* The boundary condition $u(0) = 0$ is called *essential* as it appears in the variational formulation explicitly, i.e., in the definition of V.

This type of boundary condition also frequently goes by the proper name "Dirichlet." The boundary condition $u'(1) = 0$ is called *natural* because it is incorporated implicitly. This type of boundary condition is often referred to by the name "Neumann." We summarize the different kinds of boundary conditions encountered so far, together with their various names in the following table:

Table 0.1. Naming conventions for two types of boundary conditions

Boundary Condition	**Variational Name**	**Proper Name**
$u(x) = 0$	essential	Dirichlet
$u'(x) = 0$	natural	Neumann

The assumptions $f \in C^0([0,1])$ and $u \in C^2([0,1])$ in the theorem allow (0.1.1) to be interpreted in the usual sense. However, we will see other ways in which to interpret (0.1.1), and indeed the theorem says that the formulation (0.1.3) is a way to interpret it that is valid with much less restrictive assumptions on f. For this reason, (0.1.3) is also called a *weak* formulation of (0.1.1).

0.2 Ritz-Galerkin Approximation

Let $S \subset V$ be any (finite dimensional) subspace. Let us consider (0.1.3) with V replaced by S, namely

$$(0.2.1) \qquad u_S \in S \quad \text{such that} \quad a(u_S, v) = (f, v) \qquad \forall v \in S.$$

It is remarkable that a discrete scheme for approximating (0.1.1) can be defined so easily. This is only one powerful aspect of the Ritz-Galerkin method. However, we first must see that (0.2.1) does indeed *define* an object. In the process we will indicate how (0.2.1) represents a (square, finite) system of equations for u_S. These will be done in the following theorem and its proof.

(0.2.2) Theorem. *Given $f \in L^2(0,1)$, (0.2.1) has a unique solution.*

Proof. Let us write (0.2.1) in terms of a basis $\{\phi_i : 1 \leq i \leq n\}$ of S. Let $u_S = \sum_{j=1}^n U_j \phi_j$; let $K_{ij} = a(\phi_j, \phi_i), F_i = (f, \phi_i)$ for $i, j = 1, ..., n$. Set $\mathbf{U} = (U_j), \mathbf{K} = (K_{ij})$ and $\mathbf{F} = (F_i)$. Then (0.2.1) is equivalent to solving the (square) matrix equation

$$(0.2.3) \qquad \mathbf{KU} = \mathbf{F}.$$

For a square system such as (0.2.3) we know that uniqueness is equivalent to existence, as this is a *finite dimensional* system. Nonuniqueness would imply that there is a nonzero $\mathbf{V}$ such that $\mathbf{KV} = \mathbf{0}$. Write $v = \sum V_j\phi_j$ and note that the equivalence of (0.2.1) and (0.2.3) implies that $a(v, \phi_j) = 0$ for all j. Multiplying this by V_j and summing over j yields $0 = a(v, v) = \int_0^1 (v')^2(x)\,dx$, from which we conclude that $v' \equiv 0$. Thus, v is constant, and, since $v \in S \subset V$ implies $v(0) = 0$, we must have $v \equiv 0$. Since $\{\phi_i : 1 \leq i \leq n\}$ is a basis of S, this means that $\mathbf{V} = \mathbf{0}$. Thus, the solution to (0.2.3) must be unique (and hence must exist). Therefore, the solution u_S to (0.2.1) must also exist and be unique. □

(0.2.4) *Remark.* Two subtle points are hidden in the "proof" of Theorem (0.2.2). Why is it that "thus v is constant"? And, moreover, why does $v \in V$ really imply $v(0) = 0$ (even though it is in the definition, i.e., why does the definition make sense)? The first question should worry those familiar with the Cantor function whose derivative is zero almost everywhere, but is certainly not constant (it also vanishes at the left of the interval in typical constructions). Thus, something about our definition of V must rule out such functions as members. V is an example of a *Sobolev* space, and we will see that such problems do not occur in these spaces. It is clear that functions such as the Cantor function should be ruled out (in a systematic way) as candidate solutions for differential equations since it would be a nontrivial solution to the o.d.e. $u' = 0$ with initial condition $u(0) = 0$.

(0.2.5) *Remark.* The matrix $\mathbf{K}$ is often referred to as the *stiffness* matrix, a name coming from corresponding matrices in the context of structural problems. It is clearly symmetric, since the *energy* inner-product $a(\cdot, \cdot)$ is symmetric. It is also *positive definite*, since

$$\sum_{i,j=1}^{n} k_{ij} v_i v_j = a(v, v) \quad \text{where} \quad v = \sum_{j=1}^{n} v_j \phi_j.$$

Clearly, $a(v, v) \geq 0$ for all (v_j) and $a(v, v) = 0$ was already "shown" to imply $v \equiv 0$ in the proof of Theorem 0.2.3.

0.3 Error Estimates

Let us begin by observing the fundamental *orthogonality* relation between u and u_S. Subtracting (0.2.1) from (0.1.3) implies

(0.3.1) $$a(u - u_S, w) = 0 \quad \forall w \in S.$$