

# Graduate Texts in Mathematics

**Jean-Pierre Aubin**

## **Optima and Equilibria**

**An Introduction  
to Nonlinear Analysis**

**最优和平衡**

Springer-Verlag

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Jean-Pierre Aubin

# Optima and Equilibria

An Introduction to Nonlinear Analysis

Translated from the French by Stephen Wilson  
With 28 Figures

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Reprinted in China by Beijing World Publishing Corporation, 1998

This book is dedicated to Alain Bensoussan, Ivar Ekeland, Pierre-Marie Larnac and Francine Roure, in memory of the adventure which brought us together more than twenty years ago to found the U.E.R. and the Centre de Recherche de Mathématiques de la Décision (CEREMADE).

Jean-Pierre Aubin

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Doubtless you have often been asked about the purpose of mathematics and whether the delicate constructions which we conceive as entities are not artificial and generated at whim. Amongst those who ask this question, I would single out the practical minded who only look to us for the means to make money. Such people do not deserve a reply.

Henri Poincaré  
*La Valeur de la Science*  
Chapter V

In his use of mathematical techniques to study general economic phenomena relating to countries or individuals Mr. Léon Walras has truly instituted a science.

Charles Péguy  
Un économiste socialiste, Mr. Léon Walras  
*La Revue Socialiste*, no. 146, 1897

It may be that the coldness and the objectivity for which we often reproach scientists are more suitable than feverishness and subjectivity as far as certain human problems are concerned. It is passions which use science to support their cause. Science does not lead to racism and hatred. Hatred calls on science to justify its racism. Some scientists may be reproached for the ardour with which they sometimes defend their ideas. But genocide has never been perpetrated in order to ensure the success of a scientific theory. At the end of this the XXth century, it should be clear to everyone that no system can explain the world in all its aspects and detail. Quashing the idea of an intangible and eternal truth is possibly not the least claim to fame of the scientific approach.

François Jacob  
*Le Jeu des possibles*  
Fayard (1981) p. 12

I enjoy talking to great minds and this is a taste which I like to instil in my students. I find that students need someone to admire; since they cannot normally admire their teachers because their teachers are examiners or are not admirable, they must admire great minds while, for their part, teachers must interpret great minds for their students.

Raymond Aron  
*Le Spectateur engagé*  
Julliard (1981) p. 302

# Foreword

## By Way of Warning

As in ordinary language, metaphors may be used in mathematics to explain a given phenomenon by associating it with another which is (or is considered to be) more familiar. It is this sense of familiarity, whether individual or collective, innate or acquired by education, which enables one to convince oneself that one has understood the phenomenon in question.

Contrary to popular opinion, mathematics is not simply a *richer* or *more precise* language. Mathematical reasoning is a separate faculty possessed by all human brains, just like the ability to compose or listen to music, to paint or look at paintings, to believe in and follow cultural or moral codes, etc.

But it is impossible (and dangerous) to compare these various faculties within a *hierarchical* framework; in particular, one cannot speak of the *superiority* of the language of mathematics.

Naturally, the construction of mathematical metaphors requires the autonomous development of the discipline to provide theories which may be substituted for or associated with the phenomena to be explained. This is the domain of pure mathematics. The construction of the mathematical corpus obeys its own logic, like that of literature, music or art. In all these domains, an aesthetic satisfaction is at once the objective of the creative activity and a signal which enables one to recognise successful works. (Likewise, in all these domains, fashionable phenomena – reflecting social consensus – are used to develop aesthetic criteria).

That is not all. A mathematical metaphor associates a mathematical theory with another object. There are two ways of viewing this association. The first and best-known way is to search for a theory in the mathematical corpus which corresponds as precisely as possible with a given phenomenon. This is the domain of *applied mathematics*, as it is usually understood. But the association is not always made in this way; the mathematician should not be simply a purveyor of formulae for the user. Other disciplines, notably physics, have guided mathematicians in their selection of problems from amongst the many arising and have prevented them from continually turning around in the same circle by presenting them with new challenges and encouraging them to be daring and question the ideas of their predecessors. These other disciplines may also

provide mathematicians with metaphors, in that they may suggest concepts and arguments, hint at solutions and embody new modes of intuition. This is the domain of what one might call *motivated mathematics* from which the examples you will read about in this book are drawn.

You should soon realize that the work of a *motivated* mathematician is *daring*, above all where problems from the *soft* sciences, such as social sciences and, to a lesser degree, biology, are concerned. Many hours of thought may very well only lead to the mathematically obvious or to problems which cannot be solved in the short term, while the same effort expended on a structured problem of pure or applied mathematics would normally lead to visible results.

Motivated mathematicians must possess a sound knowledge of another discipline and have an adequate arsenal of mathematical techniques at their fingertips together with the capacity to create new techniques (often similar to those they already know). In a constant, difficult and frustrating dialogue they must investigate whether the problem in question can be solved using the techniques which they have at hand or, if this is not the case, they must *negotiate* a deformation of the problem (a possible restructuring which often seemingly leads to the original model being forgotten) to produce an *ad hoc* theory which they sense will be useful later. They must convince their colleagues in the other disciplines that they need a very long period for learning and appreciation in order to grasp the language of a given theory, its foundations and main results and that the proof and application of the simplest, the most naive and the most attractive results may require theorems which may be given in a number of papers over several decades; in fact, one's comprehension of a mathematical theory is never complete. In a century when no more cathedrals are being built, but impressive skyscrapers rise up so rapidly, the profession of the motivated mathematician is becoming rare. This explains why users are very often not aware of how mathematics could be used to improve aspects of the questions with which they are concerned. When users are aware of this, the intersection of their central areas of interest with the preoccupations of mathematicians is often small – users are interested in *immediate* impacts on their problems and not in the mathematical techniques that could be used and their relationship with the overall mathematical structure.

It is these constraints which distinguish mathematicians from researchers in other disciplines who use mathematics, with a *different time constant*. It is clear that the slowness and the esoteric aspect of the work of mathematicians may lead to impatience amongst those who expect them to come up with rapid responses to their problems. Thus, it is vain to hope to *pilot* the mathematics *downstream* as those who believe that scientific development may be *programmed* (or worse still, *planned*) may suggest.

In Part I, we shall only cover aspects of pure mathematics (optimisation and nonlinear analysis) and aspects of mathematics motivated by economic theory and game theory. It is still too early to talk about *applying* mathematics to economics. Several fruitful attempts have been made here and there, but mathematicians are a long way from developing the mathematical techniques

(the domains of pure mathematics) which are best adapted to the potential applications.

However, there has been much progress in the last century since the solitary pioneer Walras, despite great opposition, dared to use the tools of mathematics in the economic domain. Brouwer, von Neumann, Kakutani, Nash, Arrow, Debreu, Scarf, Shapley, Ky Fan and many others all contributed to the knowledge you are about to share.

You will surely be disappointed by the fact that these difficult theorems have little relevance to the major problems facing mankind. But, please don't be impatient, like others, in your desire for an overall, all-embracing explanation. Professional mathematicians must be very humble and modest.

It is this modesty which distinguishes mathematicians and scientists in general from prophets, ideologists and modern system analysts. The range of scientific explanations is reduced, hypotheses must be contrasted with logic (this is the case in mathematics) or with experience (thus, these explanations must be refutable). Ideologies are free from these two requirements and thus all the more seductive.

But what is the underlying motivation, other than to contribute to an explanation of reality? We are brains which perceive the outside world and which intercommunicate in various ways, using natural language, mathematics, bodily expressions, pictorial and musical techniques, etc.

*It is the consensus on the consistency of individual perceptions of the environment, which in some way measures the degree of reality in a given social group.*

Since our brains were built on the same model, and since the ability to believe in explanations appears to be innate and universal, there is a very good chance that a social group may have a sufficiently broad consensus that its members share a common concept of reality. But prophets and sages often challenge this consensus, while high priests and guardians of the ideology tend to dogmatise it and impose it on the members of the social group. (Moreover, quite often prophets and sages themselves become the high priests and guardians of the ideology.) This continual struggle forms the framework for the history of science.

Thus, research must contribute to the evolution of this consensus, teaching must disseminate it, without dogmatism, placing knowledge in its relative setting and making you take part in man's struggle, since the day when *Homo sapiens sapiens* . . . but we do not even know what happened, we do not know when, why or how our ancestors sought to agree on their perceptions of the world to create myths and theories, when why or how they transformed their faculty for exploration into an insatiable curiosity, when, why or how mathematical faculties appeared, etc.

It is not only the utilitarian nature (in the short term) which has motivated mathematicians and other scientists in their quest. We all know that without this permanent, free curiosity there would be no technical or technological progress.

## X Foreword

Perhaps you will not use the techniques you will soon master and the results you will learn in your professional life. But the hours of thought which you will have devoted to understanding these theories will (subtly and without you being aware) shape your own way of viewing the world, which seems to be the hard kernel around which knowledge organizes itself as it is acquired. At the end of the day, it is at this level that you must judge the relevance of these lessons and seek the reward for your efforts.



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# Introduction

This is a book on nonlinear analysis and its underlying motivations in economic science and game theory. It is entitled *Optima and Equilibria* since, in the final analysis, response to these motivations consists of perfecting mechanisms for selecting an element from a given set. Such *selection mechanisms* may involve either

- *optimisation of a criterion function* defined on this set (or of several functions, in the case of multi-criterion problems in game theory),  
or
- searching in this set for an *equilibrium of a given system*.

The mathematical techniques used have their origins in what is known as nonlinear analysis, and in particular, in convex analysis.

Progress in nonlinear analysis has proceeded hand in hand with that in the theory of economic equilibrium and in game theory; there is interaction between each of these areas, mathematical techniques are applied in economic science which, in turn, motivates new research and provides mathematicians with new challenges.

In the course of the book we shall have occasion to interrupt the logical course of the exposition with several historical recollections. Here, we simply note that it was Léon Walras who, at the end of the last century, suggested using mathematics in economics, when he described certain economic agents as automata seeking to optimise evaluation functions (utility, profit, etc.) and posed the problem of economic equilibrium. However, this area did not blossom until the birth of nonlinear analysis in 1910, with Brouwer's fixed-point theorem, the usefulness of which was recognised by John von Neumann when he developed the foundations of game theory in 1928. In the wake of von Neumann came the works of John Nash, Kakutani, Aumann, Shapley and many others which provided the tools used by Arrow, Debreu, Gale, Nikaido et al. to complete Walras's construction, culminating in the 1950s in the proof of the existence of economic equilibria. Under pressure from economists, operational researchers and engineers, there was stunning progress in optimisation theory, in the area of *linear programming* after the Second World War and following the work of Fenchel, in the 1960s in convex analysis. This involved the courageous step of differentiating nondifferentiable functions, albeit in a different way and for

different reasons than in distribution theory discovered by Laurent Schwartz in the 1950s. These works provided for use of the rule hinted at by Fermat more than three hundred years ago, namely that the derivative of a function is zero at points at which the function attains its optimum, in increasingly complicated problems of the calculus of variations and optimal control theory. The 1960s also saw a re-awakening of interest in nonlinear analysis for the different problem of solving nonlinear, partial-differential equations. A profusion of new results were used to clarify many questions and simplify proofs, notably using an inequality discovered in 1972 by Ky Fan.

At the time of writing, at the dawn of the 1980s, it is appropriate to take stock and draw all this together into a homogeneous whole, to provide a concise and self-contained appreciation of the fundamental results in the areas of nonlinear analysis, the theory of economic equilibrium and game theory.

Our selection will not be to everyone's taste: it is biased. For example, in our description of the theory of economic equilibrium, we do not describe consumers in terms of their utility functions but only in terms of their demand functions. A minority will certainly hold this against us. However, conscious of the criticisms made of the present-day formalism of the Walrasian model, we propose an alternative which, like Walras, retains the explanation of prices in terms of their decentralising virtues and also admits dynamic treatment.

Our succinct introduction to game theory is not orthodox, in that we have included the theory of cooperative games in the framework of the theory of fuzzy games.

In the book we accept the shackles of the static framework that are at the origin of the inadequacies and paradoxes which serve as pretexts for rejection of the use of mathematics in economic science. J. von Neumann and O. Morgenstern were also aware of this when, in 1944, at the end of the first chapter of *Theory of Games and Economic Behaviour*, they wrote:

*'Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and, therefore, preferable. But there is ample evidence from other branches of science that it is futile to try to build one as long as the static side is not thoroughly understood...'*

*'Finally, let us note a point at which the theory of social phenomena will presumably take a very definite turn away from the existing patterns of mathematical physics. This is, of course, only a surmise on a subject where much uncertainty and obscurity prevail...'*

*'Our static theory specifies equilibria... A dynamic theory, when one is found - will probably describe the changes in terms of simpler concepts.'*

Thus, this book describes the static theory and the tool which may be used to develop it, namely nonlinear analysis.

It is only now that we can hope to see the birth of a dynamic theory calling upon all other mathematical techniques (see (Aubin and Cellina 1984) and (Aubin 1991)). But, as in the past, so too now, and in the future, the static theory must be placed in its true perspective, even though this may



mean questioning its very foundations, like March and Simon (who suggested replacing optimal choices by choices that are only satisfactory) and many (less fortunate) others. Imperfect yet perfectible, mathematics has been used to put the finishing touches to the monument the foundation of which was laid by Walras. Even if this becomes an historic monument, it will always need to be visited in order to construct others from it and to understand them once constructed.

Of course, the book only claims to present an *introduction* to nonlinear analysis which can be read by those with the basic knowledge acquired in a first-level university mathematics course. It only requires the reader to have mastered the fundamental notions of topology in metric spaces and vector spaces. Only Brouwer's fixed-point theorem is assumed.

This is a book of *motivated mathematics*, i.e. a book of mathematics motivated by economics and game theory, rather than a book of mathematics *applied* to these fields. We have included a *Foreword* to take up this issue which deals with pure, applied and motivated mathematics. In our view, this is important in order to avoid setting too great store by the importance of mathematics in its interplay with social sciences.

The book is divided into two parts. Part I describes the theory, while Part II is devoted to exercises, and problem statements and solutions. The book ends with an Appendix containing a *Compendium of Results*.

In the first three chapters, we discuss the existence of solutions minimising a function, in the general framework (Chapter 1) and in the framework of convex functions (Chapter 3). Between times, we prove the projection theorem (on which so many results in functional analysis are based) together with a number of separation theorems and we study the duality relationship between convex functions and their conjugate functions.

The following three chapters are devoted to Fermat's rule which asserts that the gradient of a function is zero at any point at which the function attains its minimum. Since convex functions are not necessarily differentiable in the customary sense, the notion of the 'differential' had to be extended for Fermat's rule to apply. The simple, but unfamiliar idea consists of replacing the concept of gradient by that of subgradients, forming a *set* called a *subdifferential*. We describe a subdifferential calculus of convex functions in Chapter 4 and in Chapter 5, we exploit Fermat's rule to characterise the solutions of minimisation problems as solutions of a many-valued equation (called an *inclusion*) or as the subdifferential of another function.

In Chapter 6, we define the notion of the generalised gradient of a locally Lipschitz function, as proposed by F. Clarke in 1975. This enables us to apply Fermat's rule to functions other than differentiable functions and convex functions. It will be useful in the study of cooperative games.

Chapters 7 and 8 are devoted to the theory of two-person games; here, we prove two fundamental minimax theorems due to von Neumann (1928) and Ky Fan (1962).