# Random Matrix Theory, Interacting Particle Systems and Integrable Systems

Percy Deift
Peter Forrester
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Random matrix theory is at the intersection of linear algebra, probability theory and integrable systems, and has a wide range of applications in physics, engineering, multivariate statistics and beyond. This volume is based on a Fall 2010 MSRI program which generated the solution of long standing questions on universalities of Wigner matrices and beta-ensembles, and opened new research directions especially in relation to the KPZ universality class of interacting particle systems, and low rank perturbations. The book contains review articles and research contributions on all these topics, in addition to other core aspects of random matrix theory such as integrability and free probability theory. It will give both established and new researchers insights into the most recent advances in field and the connections among many subfields.

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# Preface

In the spring of 1999, MSRI hosted a very successful and influential one-semester program on random matrix theory (RMT) and its applications. At the workshops during the semester, there was a sense of excitement as brand new and very recent results were reported. The goal of the 2010 Program has been to showcase the many remarkable developments that have taken place since 1999 and to spur further developments in RMT and related areas of interacting particle systems (IPS) and integrable systems (IS) as well as to highlight various applications of RMT.

One of the outputs of the 1999 program was volume 40 in the MSRI Publications series, entitled "Random matrix models and their applications". Looking back on this publication today, it is clear that this volume gave a representative snapshot of topics that were occupying the attention of researchers in the field then. Moreover, the papers — consisting of a mix of research articles and reviews — provide a conveniently bundled resource for researchers in the field to this day.

Since 1999 random matrix theory has captured the imagine of a whole new generation of researchers, and through a collective effort some outstanding questions have been settled, and new highly promising research areas initiated. One example of the former is work on universality questions for Wigner matrices, where the task is to show that for large dimension a symmetric matrix with independent entries of mean zero and standard deviation 1 has the same statistical properties as in the case of standard Gaussian entries. Another is universality questions for $\beta$ ensembles, where one wants to show that the statistical properties are independent of the one-body potential. New research areas include the KPZ equation and related growth processes, which has led to the precise experimental realization of some random matrix distributions, and also to quite spectacular theoretical advances relating to a rigorous understanding of the replica trick via so called Macdonald processes; analytic predictions of the $\beta$-generalization of the so-called Dyson constant in the asymptotic expansion of spacing distributions in $\beta$-ensembles; and stochastic differential equations and PDEs for eigenvalue distributions in the case of a low rank perturbation leading to eigenvalue separation.

A core aim of the 2010 semester was to spur further developments in RMT and the related areas of interacting particle systems and integrable systems. It is our hope that this new MSRI Publications volume based on the 2010 semester will lend weight to this cause. Each author was a participant of the semester. Articles on all topics nominated above relating to solutions of outstanding questions and new research areas can be found: universality for Wigner matrices (Tao and Vu), universality for $\beta$ ensembles (Borodin, Shcherbina); KPZ equation (Quastel, Sasamoto, Spohn, Takeuchi), Macdonald process (O'Connell), Dyson constant and asymptotics of spacing distributions (Forrester), low rank perturbations (Baik and Wang, Maida). One should also highlight the work on RMT and numerical algorithms by Pfrang, Deift and Menon, which is in the spirit of one of the very early uses of random matrices by von Neumann and co-workers at the dawn of the computer era, and the extensive review of free probability theory by Novak, the latter being a write up of a series of lectures he delivered during the semester.

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# Universality conjecture for all Airy, sine and Bessel kernels in the complex plane

GERNOT AKEMANN AND MICHAEL J. PHILLIPS

We address the question of how the celebrated universality of local correlations for the real eigenvalues of Hermitian random matrices of size $N \times N$ can be extended to complex eigenvalues in the case of random matrices without symmetry. Depending on the location in the spectrum, particular large-$N$ limits (the so-called weakly non-Hermitian limits) lead to one-parameter deformations of the Airy, sine and Bessel kernels into the complex plane. This makes their universality highly suggestive for all symmetry classes. We compare all the known limiting real kernels and their deformations into the complex plane for all three Dyson indices $\beta = 1, 2, 4$, corresponding to real, complex and quaternion real matrix elements. This includes new results for Airy kernels in the complex plane for $\beta = 1, 4$. For the Gaussian ensembles of elliptic Ginibre and non-Hermitian Wishart matrices we give all kernels for finite $N$, built from orthogonal and skew-orthogonal polynomials in the complex plane. Finally we comment on how much is known to date regarding the universality of these kernels in the complex plane, and discuss some open problems.

## 1. Introduction

The topic of universality in Hermitian random matrix theory (RMT) has attracted a lot of attention in the mathematics community recently, particularly in the context of matrices with elements that are independent random variables, as reviewed in [Tao and Vu 2012; Erdős and Yau 2012]. The question that one tries to answer is this: under what conditions are the statistics of eigenvalues of $N \times N$ matrices with independent Gaussian variables the same (for large matrices) as for more general RMT where matrix elements may become coupled? This has been answered under very general assumptions, and we refer to some recent reviews on invariant [Kuijlaars 2011; Deift and Gioev 2009] and noninvariant [Tao and Vu 2012; Erdős and Yau 2012] ensembles.

In this short note we would like to advocate the idea that non-Hermitian RMT with eigenvalues in the complex plane also warrants the investigation of universality. Apart from the interest in its own right, these models have important applications in physics and other sciences (see, e.g., [Akemann et al. 2011a]). We will focus here on RMT that is close to Hermitian, a regime that is particularly important for applications in quantum chaotic scattering (see [Fyodorov and Sommers 2003] for a review) and quantum chromodynamics (QCD), for example. In the latter case, the non-Hermiticity may arise from describing the effect of quark chemical potential (as reviewed in [Verbaarschot 2011; Akemann 2007]), or from finite lattice spacing effects of the Wilson–Dirac operator (see [Damgaard et al. 2010; Akemann et al. 2011b] as well as [Kieburg 2012] for the solution of this non-Hermitian RMT).

Being a system of $N$ coupled eigenvalues, Hermitian RMT already offers a rich variety of large-$N$ limits, where one has to distinguish the bulk and (soft) edge of the spectrum for Wigner–Dyson (WD) ensembles, and in addition the origin (hard edge) for Wishart–Laguerre (WL, or chiral) RMT. Not surprisingly complex eigenvalues offer even more possibilities. The limit we will investigate is known as the weakly non-Hermitian regime; it *connects* Hermitian and (strongly) non-Hermitian RMT, and was first introduced in [Fyodorov et al. 1997a; 1997b] in the bulk of the spectrum. For strong non-Hermiticity — which includes the well-known circular law and the corresponding universality results — we refer to [Khoruzhenko and Sommers 2011] and references therein, although the picture there is also far from being complete; an important breakthrough was published recently in [Tao and Vu 2014].

In the next section we give a brief list of the six non-Hermitian WD and WL ensembles, and indicate where they were first solved in the weak limit. There are three principal reasons why we believe that universality may hold. First, in some cases two different Gaussian RMT both give the same answers. Second, there are heuristic arguments available for i.i.d. matrix elements using supersymmetry [Fyodorov et al. 1998], as well as for invariant non-Gaussian ensembles using large-$N$ factorisation and orthogonal polynomials (OP) [Akemann 2002]. Third, the resulting limiting kernels of (skew-) OP look very similar to the corresponding kernels of real eigenvalues, being merely one-parameter deformations of them. One of the main goals of this paper is to illustrate this fact. For this purpose we give a complete list of all the known Airy, sine, and Bessel kernels for real eigenvalues, side-by-side with their deformed kernels in the complex plane, where some of our results are new.

## 2. Random matrices and their limiting kernels

In this section we briefly introduce the Gaussian random matrix ensembles that we consider, and give a list of the limiting kernels they lead to, for both real and complex eigenvalues. For simplicity we have restricted ourselves to Gaussian ensembles in the Hermitian cases, in order to highlight the parallels to their non-Hermitian counterparts.

We begin with the classical WD and Ginibre ensembles in Section 2.1, displaying Airy (Section 2.2) and sine (Section 2.3) behaviour at the (soft) edge and in the bulk of the spectrum respectively, as well as their deformations. We then introduce the WL ensembles and their non-Hermitian counterparts in Section 2.4, in order to access the Bessel behaviour (Section 2.5) at the origin (or hard edge). The corresponding orthogonal and skew-orthogonal Hermite and Laguerre polynomials are given in Appendix A, and precise statements of the limits that lead to the microscopic kernels can be found in Appendix B.

**2.1.** ***Gaussian ensembles with eigenvalues on*** $\mathbb{R}$ ***and*** $\mathbb{C}$. The three classical Gaussian Wigner–Dyson ensembles (the GOE, GUE and GSE) are defined as [Mehta 2004]

$$\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}} = \int dH \exp[-\beta \operatorname{Tr} H^2/4] = c_{N,\beta} \prod_{j=1}^{N} \int_{\mathbb{R}} dx_j \, w_\beta(x_j) \, |\Delta_N(\{x\})|^\beta. \tag{2.1}$$

The random matrix elements $H_{kl}$ are real, complex, or quaternion real numbers for $\beta = 1, 2, 4$ respectively, with the condition that the $N \times N$ matrix $H$ ($N$ is taken to be even for simplicity) is real symmetric, complex Hermitian or complex Hermitian and self-dual for $\beta = 1, 2, 4$. In the first equation we integrate over all independent matrix elements denoted by $dH$. The Gaussian weight completely factorises and thus the independent elements are normal random variables; for $\beta = 1$, for example, the real elements are distributed $\mathcal{N}(0, 1)$ for off-diagonal elements, and $\mathcal{N}(0, 2)$ for diagonal elements.

In the second equality of (2.1), we diagonalised the matrix

$$H = U \operatorname{diag}(x_1, \ldots, x_N) \, U^{-1},$$

where $U$ is an orthogonal, unitary or unitary-symplectic matrix for $\beta = 1, 2, 4$. The integral over $U$ factorises and leads to the known constants $c_{N,\beta}$. We obtain a Gaussian weight $w_\beta(x)$ and the Vandermonde determinant $\Delta_N(\{x\})$ from the Jacobian of the diagonalisation,

$$w_\beta(x) = \exp[-\beta x^2/4], \quad \Delta_N(\{x\}) = \prod_{1 \le l < k \le N} (x_k - x_l). \tag{2.2}$$

The integrand on the right-hand side of (2.1) times $c_{N,\beta}/\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}}$ defines the normalised joint probability distribution function (jpdf) of all eigenvalues. The $k$-point correlation function $R_k^\beta$, which is proportional to the jpdf integrated over

$N-k$ eigenvalues, can be expressed through a single kernel $K_N^{\beta=2}$ of orthogonal polynomials (OP) for $\beta=2$, or through a $2\times 2$ matrix-valued kernel involving skew-OP for $\beta=1,4$:

$$\begin{aligned} R_k^{\beta=2}(x_1,\ldots,x_k) &= \det_{i,j=1,\ldots,k}[K_N^{\beta=2}(x_i,x_j)], \\ R_k^{\beta=1,4}(x_1,\ldots,x_k) &= \operatorname*{Pf}_{i,j=1,\ldots,k}\left[\begin{pmatrix} K_N^{\beta=1,4}(x_i,x_j) & -G_N^{\beta=1,4}(x_i,x_j) \\ G_N^{\beta=1,4}(x_j,x_i) & -W_N^{\beta=1,4}(x_i,x_j)\end{pmatrix}\right]. \end{aligned} \tag{2.3}$$

The matrix kernel elements $K_N$ and $W_N$ are not independent of $G_N$ but are related by differentiation and integration, respectively. These relations will be given later for the limiting kernels.

The three parameter-dependent Ginibre (i.e., elliptic or Ginibre–Girko) ensembles, denoted by GinOE, GinUE, and GinSE, can be written as

$$\begin{aligned} \mathcal{Z}_N^{\mathrm{Gin}\beta\mathrm{E}}(\tau) &= \int dJ \exp\left[\frac{-\gamma_\beta}{1-\tau^2}\operatorname{Tr}\left(JJ^\dagger - \frac{\tau}{2}(J^2+J^{\dagger 2})\right)\right] \\ &= \int dH_1\, dH_2 \exp\left[-\frac{\gamma_\beta \operatorname{Tr} H_1^2}{1+\tau} - \frac{\gamma_\beta \operatorname{Tr} H_2^2}{1-\tau}\right], \end{aligned} \tag{2.4}$$

with $\tau\in[0,1)$. We use the parametrisation of [Khoruzhenko and Sommers 2011], with $\gamma_{\beta=2}=1$ and $\gamma_{\beta=1,4}=\frac{1}{2}$. The matrix elements of $J$ are of the same types as for $H$ for all three values of $\beta$, but without any further symmetry constraint. Decomposing $J=H_1+iH_2$ into its Hermitian and anti-Hermitian parts, these ensembles can be viewed as Gaussian two-matrix models. For $\tau=0$ (maximal non-Hermiticity) the distribution for all matrix elements again factorises. In the opposite, that is, Hermitian, limit ($\tau\to 1$), the parameter-dependent Ginibre ensembles become the Wigner–Dyson ensembles. The jpdf of complex (and real) eigenvalues can be computed by transforming $J$ into the following form, $J=U(Z+T)U^{-1}$. For $\beta=2$ this is the Schur decomposition, with $Z=\operatorname{diag}(z_1,\ldots,z_N)$ containing the complex eigenvalues, and $T$ being upper triangular:[1]

$$\begin{aligned} \mathcal{Z}_N^{\mathrm{GinUE}}(\tau) &= c_{N,\mathbb{C}}^{\beta=2}\prod_{j=1}^N \int_{\mathbb{C}} d^2z_j\, w_{\beta=2}^{\mathbb{C}}(z_j)\, |\Delta_N(\{z\})|^2, \\ w_{\beta=2}^{\mathbb{C}}(z) &= \exp\left[\frac{-1}{1-\tau^2}\left(|z|^2-\frac{\tau}{2}(z^2+z^{*2})\right)\right]. \end{aligned} \tag{2.5}$$

For $\beta=1,4$ we follow [Khoruzhenko and Sommers 2011] where the two ensembles have been cast into a unifying framework. For simplicity we choose

[1] The resulting jpdf of complex eigenvalues for normal matrices with $T\equiv 0$ at $\beta=2$ is the same.

$N$ to be even. Here the matrix $Z$ can be chosen to be $2\times 2$ block diagonal and $T$ to be upper block triangular. The calculation of the jpdf reduces to a $2\times 2$ calculation, yielding

$$\mathcal{Z}_N^{\text{GinO/SE}}(\tau) = c_{N,\mathbb{C}}^{\beta=1,4} \prod_{j=1}^{N} \int_{\mathbb{C}} d^2 z_j \prod_{k=1}^{N/2} \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_{2k-1}, z_{2k})\, \Delta_N(\{z\}), \tag{2.6}$$

where we have introduced an antisymmetric bivariate weight function. For $\beta=1$, this is given by

$$\begin{aligned} \mathcal{F}_{\beta=1}^{\mathbb{C}}(z_1, z_2) &= w_{\beta=1}^{\mathbb{C}}(z_1) w_{\beta=1}^{\mathbb{C}}(z_2) \\ &\qquad \times \big(2i\delta^2(z_1 - z_2^*)\,\text{sign}(y_1) + \delta^1(y_1)\delta^1(y_2)\,\text{sign}(x_2 - x_1)\big), \end{aligned}$$

$$(w_{\beta=1}^{\mathbb{C}}(z))^2 = \text{erfc}\left(\frac{|z - z^*|}{\sqrt{2(1-\tau^2)}}\right) \exp\left[\frac{-1}{2(1+\tau)}(z^2 + z^{*2})\right], \tag{2.7}$$

and for $\beta=4$ by

$$\begin{aligned} \mathcal{F}_{\beta=4}^{\mathbb{C}}(z_1, z_2) &= w_{\beta=4}^{\mathbb{C}}(z_1) w_{\beta=4}^{\mathbb{C}}(z_2)\,(z_1 - z_2)\,\delta(z_1 - z_2^*), \\ (w_{\beta=4}^{\mathbb{C}}(z))^2 &= w_{\beta=2}^{\mathbb{C}}(z). \end{aligned} \tag{2.8}$$

For $\beta=1$, it should be noted that the integrand in (2.6) is not always positive, and so a symmetrisation must be applied when determining the correlation functions below.[2] For $\beta=4$, the parameter $N$ in (2.6) should — in our convention — be taken to be the size of the complex-valued matrix that is equivalent to the original quaternion real matrix.

The correlation functions can be written in a similar form as for the real eigenvalues

$$\begin{aligned} R_{k,\mathbb{C}}^{\beta=2}(z_1, \ldots, z_k) &= \det_{i,j=1,\ldots,k} [K_{N,\mathbb{C}}^{\beta=2}(z_i, z_j^*)], \\ R_{k,\mathbb{C}}^{\beta=1,4}(z_1, \ldots, z_k) &= \operatorname*{Pf}_{i,j=1,\ldots,k}\left[\begin{pmatrix} K_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) & -G_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) \\ G_{N,\mathbb{C}}^{\beta=1,4}(z_j, z_i) & -W_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) \end{pmatrix}\right], \end{aligned} \tag{2.9}$$

where the elements of the matrix kernels are related through

$$\begin{aligned} G_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) &= -\int_{\mathbb{C}} d^2 z\, K_{N,\mathbb{C}}^{\beta=1,4}(z_i, z) \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z, z_j), \\ W_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) &= \int_{\mathbb{C}^2} d^2 z\, d^2 z'\, \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i, z) K_{N,\mathbb{C}}^{\beta=1,4}(z, z') \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z', z_j) \\ &\qquad - \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i, z_j). \end{aligned} \tag{2.10}$$

[2]It is, however, possible to write the partition function $\mathcal{Z}_N^{\text{GinOE}}$ as an integral over a true (i.e., positive) jpdf, by, for example, appropriately ordering the eigenvalues; however, such a representation is technically more difficult to work with.

The kernels $K^{\beta}_{N,\mathbb{C}}(z,z')$ are given explicitly in Appendix A.

For $\beta=1$, we can write

$$\begin{aligned} G^{\beta=1}_{N,\mathbb{C}}(z_1,z_2) &= \delta^1(y_2)G^{\beta=1}_{N,\mathbb{C},\mathrm{real}}(z_1,x_2)+G^{\beta=1}_{N,\mathbb{C},\mathrm{com}}(z_1,z_2),\\ W^{\beta=1}_{N,\mathbb{C}}(z_1,z_2) &= \delta^1(y_1)\delta^1(y_2)W^{\beta=1}_{N,\mathbb{C},\mathrm{real,real}}(x_1,x_2)\\ &\quad+\delta^1(y_1)W^{\beta=1}_{N,\mathbb{C},\mathrm{real,com}}(x_1,z_2)+\delta^1(y_2)W^{\beta=1}_{N,\mathbb{C},\mathrm{com,real}}(z_1,x_2)\\ &\quad+W^{\beta=1}_{N,\mathbb{C},\mathrm{com,com}}(z_1,z_2)-\mathcal{F}^{\mathbb{C}}_{\beta=1}(z_1,z_2), \end{aligned} \tag{2.11}$$

whereas, for $\beta=4$, (2.8) implies the following relations:

$$\begin{aligned} G^{\beta=4}_{N,\mathbb{C}}(z_1,z_2) &= (z_2-z_2^*)w^{\mathbb{C}}_{\beta=2}(z_2)K^{\beta=4}_{N,\mathbb{C}}(z_1,z_2^*),\\ W^{\beta=4}_{N,\mathbb{C}}(z_1,z_2) &= -(z_1-z_1^*)(z_2-z_2^*)w^{\mathbb{C}}_{\beta=2}(z_1)w^{\mathbb{C}}_{\beta=2}(z_2)K^{\beta=4}_{N,\mathbb{C}}(z_1^*,z_2^*), \end{aligned} \tag{2.12}$$

where in the final expression we have dropped the term representing the perfect correlation between an eigenvalue $z$ and its complex conjugate $z^*$. For this reason, for $\beta=4$ we will only give one of the matrix kernel elements in the following.

Note that $\beta=1$ is special as the eigenvalues of a real asymmetric matrix are either real or come in complex conjugate pairs. Therefore we will have to distinguish kernels (and $k$-point densities) of real, complex or mixed arguments.

In order to specify the limiting kernels we first need the behaviour of the mean (or macroscopic) spectral density. At large $N$, and for all three values of $\beta$, the (real) eigenvalues in the Hermitian ensembles are predominantly concentrated within the Wigner semicircle $\rho_{\mathrm{sc}}(x)=(2\pi N)^{-1}\sqrt{4N-x^2}$ on $[-2\sqrt{N},2\sqrt{N}]$, whereas in the non-Hermitian ensemble, the complex eigenvalues lie mostly within an ellipse with half-axes of lengths $(1+\tau)\sqrt{N}$ and $(1-\tau)\sqrt{N}$, with constant density $\rho_{\mathrm{el}}(z)=(N\pi(1-\tau^2))^{-1}$. Depending on where (and how) we magnify the spectrum locally, we obtain different asymptotic Airy or sine kernels for each $\beta=1,2,4$. In the following we will give all of the known real kernels; see [Kuijlaars 2011], for example, for a complete list and references, together with their deformations into the complex plane. For the Bessel kernels which will be introduced later we need to consider different matrix ensembles, see Section 2.4 below.

**2.2.** ***Limiting Airy kernels on $\mathbb{R}$ and $\mathbb{C}$.*** When appropriately zooming into the "square root" edge of the semicircle, the three well-known Airy kernels (matrix-valued for $\beta=1,4$) are obtained for real eigenvalues. For complex eigenvalues we have to consider the vicinity of the eigenvalues on a thin ellipse which have the largest real parts, and where the weakly non-Hermitian limit introduced in [Bender 2010] is defined such that

$$\sigma=N^{\frac{1}{6}}\sqrt{1-\tau} \tag{2.13}$$