CALCULUS

WITH RELATED TOPICS

SUN, JIA YONG

高等学校教材

Calculus with Related Topics

微积分学

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Calculus with Related Topics (First Edition)

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内 容 简 介

本书以英文撰写，从1985年起已在西北工业大学使用过三遍。它包含了工科院校高等数学教学基本要求的全部内容并增加了少量有用的内容，如矢值函数、含参变量的积分等。本书着重于论理并注意更新内容。它以勒贝格积分代替黎曼积分，以L^2中的付立叶级数理论代替了古典理论，其它的讲法和体系也有较大变动。

本书共12章。前8章属一元微分学范畴，可于第一学期以96学时讲完；后4章是一元积分学，无穷级数及多元微积分学可于第二学期以112学时讲完。

本书供程度较高的理、工科大学一年级学生使用。

Preface

This book has been used in NPU for 3 times since 1985.

It contains all the mathematical materials usually taught for freshmen in China and supplies a few extra useful materials, such as vector valued functions and integrals with a parameter.

The main difference between this book and the others is that theoretical deductions are enhanced in this book. In this book the theory of Riemann integral is replaced by the theory of Lebesgue integral, the classical theory of Fourier series is replaced by the L^2 theory, etc.

There are 12 chapters in this book. The first 8 chapters deal with materials concerning the differential calculus of functions of one variable and will be covered in the first semester with 96 hrs.(80 lecture hrs. and 16 practice hrs.). The remaining 4 chapters deal with the integral calculus of functions of one variable, infinite series and the calculus of functions of several variables and will be covered in the second semester with 112 hrs.(96 lecture hrs. and 16 practice hrs.)

This book is supplied for freshmen in a higher level.

Jiayong Sun

July, 1988

Foreword

Calculus was invented by Newton and Leibniz independently in the 17th century. Using Calculus Newton solved some problems in Dynamics, and Leibniz, in Geometry. Thereafter, a great progression in Calculus has been made by many famous mathematicians, e.g. Euler, Cauchy, Riemann, Lagrange, Weierstrass and others. It is now well known that most of the natural laws must be described and investigated by Calculus. So Calculus is very important for scientists and engineers.

Calculus consists of two essential parts, the differential calculus ane the integral calculus. The theory of differential calculus discussed here is classical, but the theory of integral calculus is modern, which is achieved by Lebesgue and other mathematicians in this century.

We will discuss the theory and application of Calculus for functions of one variable at first and then functions of several variables. Some closely related topics, such as elementary differential equations, infinite series and vectors are inserted appropriately.

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Chapter 1 Functions

Function is the main object Calculus deals with. In this chapter we will introduce the concept of a function and some other basic concepts.

1.1 Real numbers and the number axis

1° Real numbers

By real numbers we mean the terminating or non-terminating decimals. $\frac{1}{2}$, $-\frac{1}{3}$, $\sqrt{2}$, π, ··· are real numbers, since they can be converted into decimals. Hereafter we will mainly deal with real numbers and call them numbers for simplicity.

2° The number axis

A number axis is a directed line with a fixed point O on it and with a unit length attached.

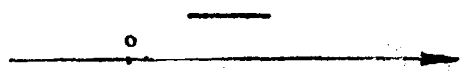


fig 1.1.1

It is well known that every number corresponds to one and only one point on the number axis, and conversly. Because of this fact, it will cause no confusion, if we don't distinguish the term point and number rigorously.

3° $+\infty$ (positive infinity)

$+\infty$ is an ideal number which is thought to be greater than any real number x, ($+\infty > x$). It corresponds to the point located at infinity on the right hand side of the number axis. It should be noted that $+\infty$ is not a real number, so any arithmetical operation about $+\infty$ is not yet defined.

4° $-\infty$ (negative infinity)

$-\infty$ may be discussed in a similar way.

1.2 Number sets and bounds of sets

1° Number sets

By a number set, or simply set, we mean an aggregate of real numbers with certain property.

Ex.1 The aggregate of all those x, such that $0<x<3$ holds is a set.

Ex.2 The aggregate of all those x, such that $x^2-1=0$ holds is a set.

The set of all those x, such that property p holds for them, is denoted by $\{x|p\}$. According to this notation, the sets in the above examples may be written as $\{x|0<x<3\}$ and $\{x|x^2-1=0\}$, respectively.

Sometimes, a set may consist of no number. For example, $\{x|x^2<0\}$ is such a set, since $x^2 \geqslant 0$ for any real number x. A set consists of no number is called an empty set or null set, and denoted by ϕ. A set consists of all the numbers is denoted by R.

2° Bounds of sets

Let S be a given set and M be a number, such that $M \geqslant x$, for any x belongs to S, or symbolically $M \geqslant x$, $\forall\, x \in S$, ("$\forall$" means "for any" and "$\in$", "belongs to")

We call such M an upper bound of S. And S is said to be bounded above (by M).

For example, let $S = \{x \mid 0 < 3\}$ and $M = 4$, then $4 \geqslant x$, $\forall\, x \in S$. So 4 is an upper bound of this set S and S is bounded above. Similarly, 5, 6, 7.2, ··· are all upper bounds of S. Among these upper bounds, there is a least one, 3, which is called the least upper bound of S.

The least upper bound of S may or may not belong to S. For example, the least upper bound of the set S in Ex.1 does not belong to S while the least upper bound of the set S in Ex.2 does, etc.

It is an important property that if a non-empty set S has an upper bound, then it has a least upper bound. This important property is called the Dedkind's property.

The least upper bound of S is denoted by l.u.b. S or sup S (Supremum of S).

It should be noted that if the set has no upper bound at all, then it certainly has no least upper bound. For example, the set J consists of all positive integers has no upper bound and hence no least upper bound.

If a set S has no upper bound, we say that l.u.b. S or sup S is $+\infty$.

The lower bound and greatest lower bound of a set may be discussed in a similar way and we leave it as an exercise.

The greatest lower bound of S is denoted by g.l.b.S or inf S (infrmum of S).

The sets bounded above and below are called bounded sets.

1.3 Intervals and neighborhoods

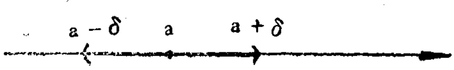
1° Intervals

The set $\{x \mid x$ lies between a and b$\}$ is called an interval a, b. It may be represented by a segment on the number axis, a and b are called the boundary points of this interval, which may or may not be included in the interval. When a and b are included in the interval, the interval is called a closed interval. i.e. a closed interval is the set $\{x \mid a \leqslant x \leqslant b\}$. we denote this closed interval by $[a, b]$. when a and b are not included in the interval, the interval is called an open interval, i.e. an open interval is the set $\{x \mid a < x < b\}$. we denote this open interval by (a, b). The interval a, b, which includes just one boundary point is called a half open interval. A half open interal a, b is denoted by $[a, b)$ or $(a, b]$ acoording as a or b is included. We use the square or the round bracket to distinguish the boundary point is included or not. In case we do not care about whether the boundary points are included or not, we just say the interval a, b, and denote it by $\langle a, b\rangle$. The points other than the boundary points are called the interior points of the interval which have the property that each of them has a neighborhood contained entirely in the interval, as will be seen in the next article.

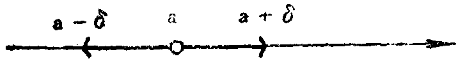
2° Neighborhoods

$(a-\delta, a+\delta)$, where δ is any positive number, is called a neighborhood of a,

with radius δ, which is an open interval with center a and radius δ.
$(a-\delta,\ a+\delta)$, with a excluded, is called a deleted neighborhood of a.



neighborhood
fig 1.3.1

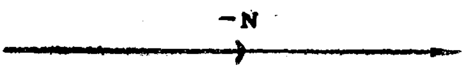


deleted neighborhood
fig 1.3.2

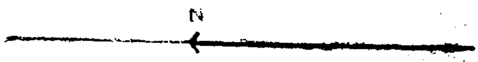
Ex.1 $(1-2,\ 1+2)$ is a neighborhood of 1 (with radius 2);
$(1-2,\ 1+2)$, with 1 excluded, is a deleted neighborhood of 1.

Any number has infinitely many neighborhoods.

The set $\{x \mid x > N\}$ is called a (deleted) neighborhood of $+\infty$; the set $\{x \mid x < -N\}$ is called a (deleted) neighborhood of $-\infty$.



neighborhood of $+\infty$
fig 1.3.3



neighborhood of $-\infty$
fig 1.3.4

The intuitive meaning of neighborhorhoods is obvious.

Sometimes we also call $[a,\ a+\delta)$ and $(a-\delta,\ a]$ the right and left neighborhood of a, respectively.

1.4 Variables and functions

1° Variables

Quantities that may take various values, occur frequently in our daily life. we call such quantities the variables.

Ex.1 If a car is being driven then the time t (measured in minutes) of movement is a variable, the distance d (measured in miles) being covered by the car is a variable, the temperature T (measured in ℃)in the car is a variable, etc.

Ex.2 If a gold ring is being heated, then the radius r (measured in cm.) of the ring is a variable, the circumstance c (measured in cm.) of the ring is a variable.

2° Functions

A quantity is called a function of another variable, defined on some set D, iff (if and only if)the value of this quantity is uniquely determined by the value of the other variable taken from D. We call this variable the independent variable,and D the domain of the function.

Ex.3 $3x+2$ is a function of the variable x, defined on R, since the value of $3x+2$ is uniquely determined by the value of x taken from R.

Ex.4 x^2 is a function of the variable x, defined on R.

Ex.5 When the velocity of a moving car is given,the distance d (measured in miles) being covered by the car is a function of the time t (measurel in minutes) of movement, defined on some interval $[0,\ N]$.

Ex.6 In Ex.2, the circumference c is a function of r defined on some interval(L, M).

A function of the variable x is denoted by f(x) or g(x), φ(x), ··· . If the function is defined on D, we may add a supplement that it is defined on D. We must use different symbols to denote different functions.

The value of a function f(x), as x takes the value a is denoted by $f(x)\big|_{x=a}$ or f(a). If f(a)=0, then a is said to be a zero of f(x).

Ex.7 If f(x)=3x+2, then $f(x)\Big|_{x=-\frac{2}{3}}=(3x+2)\Big|_{x=-\frac{2}{3}}=3\left(\frac{-2}{3}\right)+2=0$. That is,

$-\frac{2}{3}$ is a zero of 3x+2.

Sometimes we denote a specified value taken by the variable x by the symbol x as well, but distinguish them by the words "variable" and "value" , if necessary. Accordingly, f(x) is a function of x, if x denotes a variable; f(x) is a value of a function, if x denotes a value.

3° Ranges and graphs

By the range of a function f(x) defined on D, we mean the set of values of f(x), as x takes all values over D.

Ex.8 The range of 3x+2 is R, since $\forall$ a $\in$R, there is an x $\in$ R such that 3x+2 =a. The range of x^{-2} is the positive real number set R^+, since $\forall$ a $\in R^+$, there is x $\in$ R, such that x^{-2}=a and for non-positive a, there isn't.

By the graph of a function f(x), defined on D we mean the set of all points (x, y), with y=f(x) and x takes values over D.

Ex.9 The graph of the function 3x+2 defined on R. is the set of all (x, y) with y=3x+2 and x takes values over R.

Thus (−1, −1), (0, 2), (1, 5), ··· are all points of the graph.The whole graph is a straight line on the xy-plane.

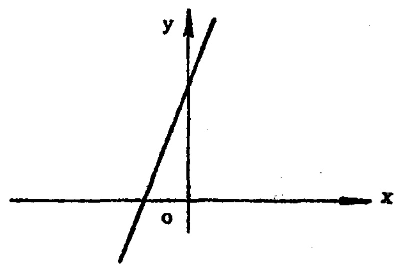


fig 1.3.5

Generally, we can get the graph of a function approximately by plotting many points of the graph. The more points we plot, the more accurate figure we get.

4° A note about the domain

The domain of a function is usually not mentioned in the following two cases:

1) It is clear from the context.

Ex.10 In Ex.5, we may just say that d is a function of t ; In Ex.6, we may just say that c is a function of r.

2) It is clear from the expression.

Why? In mathematics we have the convention that if the expression of f(x) is given, then its domain is always assumed to be the set of all those values of x for which the expxession has a meaning, unless otherwise specified.

Ex.11 If $f(x)=\frac{1}{\sqrt{x-1}}$, find its domain.

By convention, we have to find out the set of all those x, for which the expression $\frac{1}{\sqrt{x-1}}$ has a meaning. we try to do this.

Since we only consider real numbers, the number under the radical sign "$\sqrt{\quad}$" must $\geqslant 0$, otherwise the expression has no meaning,

$$x-1 \geqslant 0, \text{ i.e. } x \geqslant 1$$

Moreover, since the denominator of a fraoction oan never be 0, otherwise the fraction is meaningless, so

$$x-1 \neq 0 \quad \text{i.e.} \quad x \neq 1$$

Combining these two results, we get the set of all those x for which $\frac{1}{\sqrt{x-1}}$ has a meaning is, $\{x|x>1\}$ i.e $(1, +\infty)$. This is the domain of the given function.

1.5 To express functions

1° Some examples

In the last section, we have seen many examples of functions. In Ex.3 and Ex.4, $3x+2$ and x^2 are functions of x; In Ex.5 and Ex.6 d is a function of t and c is a function of r. There is a main difference between these functions. In Ex.3 and Ex.4, how to get the value of the function from the value of the independent variable is known, while in Ex.5 and Ex.6 is not. A function is said to be expressed, if how to get its value from the value of the independent variable is known. Otherwise, the function is said to be unexpressed. How to make an unexpressed function expressed is important in application. If we can not make a function expressed, then we are usually unable to investigate it more concretely. But there is no general rule to make a function expressed. We must use different rules for different problems accordingly. The following are some examnles.

Ex.1 Make the unexpressed functions disoussed above expressed.

Solution. In Ex.5, 1.4, by physics,

$d = v_0 t$ (Suppose the car moves with a constant speed v_0)

In Ex.6, 1.4, by Geometry, $c = 2\pi r$.

Ex.2 Enclose a reotangle by a string of a given length 1. Its area A is a function of its width x. Make it expressed (or express A as a function of x).

Solution. We want to find the value of A for any specified value x. Draw a figure. Let x be the width, h the height. From the figure, $A = xh$.

But

$$1 = 2x + 2h$$

So

$$h = \frac{1}{2}(1-x).$$

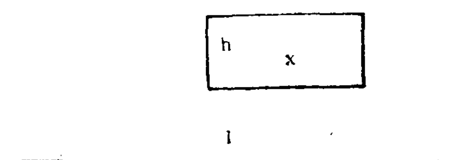


fig 1.5.1

Hence

$$A = \frac{1}{2}(1-2x)x$$

Ex.3 Mr. A is 20 ml. due north of Mr. B

A walks to the east with a velocity 6 ml/hr;

B walks to the north with a velocity 5 ml/hr.

After t hrs. the distance between A and B is d miles. d is a function of t. Make it expressed. (or express d as a function of t)

Solution. We want to find the value d for any specified value t. Draw a figure at first. Originally, Mr. A and Mr. B are at (A) and (B) respectively. After t hrs. They arrive at A and B respectively.

From the figure,

$$d^2 = (BA)^2$$
$$= |B(A)|^2 + |(A)A|^2$$
$$= (20-5t)^2 + (6t)^2$$

by Pythagorean formula for right triangles.

Thus

$$d = \pm\sqrt{(20-5t)^2 + (6t)^2}$$

But $d \geqslant 0$, so

$$d = \sqrt{(20-5t)^2 + (6t)^2}$$

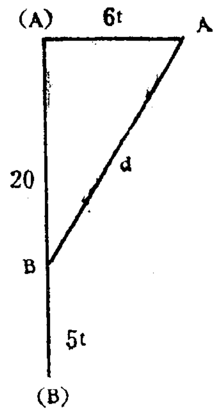


fig 1.5.2

Ex.4 Consider a variable rectangle inscribed in a given circle of radius R, i.e. a variable rectangle with vertices on the circle. It's area A is a function of its width x. Make it expressed (or express A as a function of x)

Solution. Draw a figure to make the meaning clear. From the figure,

$$A = xl$$

But by Pythagorean formula for right triangles

$$l^2 = (2R)^2 - x^2$$

i.e.

$$l = \sqrt{(2R)^2 - x^2}$$

So

$$A = x\sqrt{(2R)^2 - x^2}$$

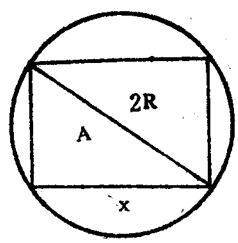


fig 1.5.3

Ex.5 An open rectangular box is to be made from a piece of cardboard 8cm. wide

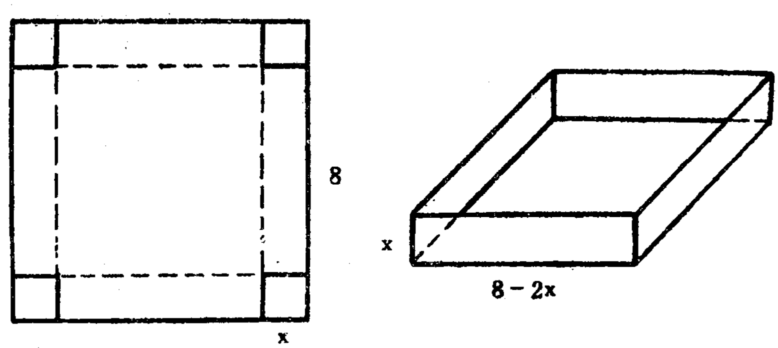


fig 1.5.4

and 8cm. long by cutting a square from each corner and bending up the sides. Its volume V is a function of the side x of the square cut. Express this function.

Solution. Draw the necessary figures, From the figures,

$$V=(8-2x)^2x$$

Ex.6 The length 1 of the chord suspended by an arc on a given circle of radius R is a function of the length x of the arc. Express it.

Solution. Draw a figure. From the figure, we see that the central angle suspended by the arc is $\frac{x}{R}$ radians. Hence the length 1 of the chord is

$$1=2R\sin\frac{1}{2}\left(\frac{x}{R}\right)=2R\ \sin\frac{x}{2R}$$

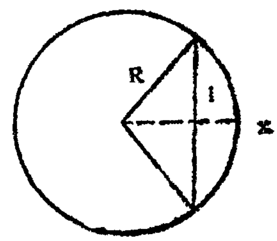


fig 1.5.5

Ex.7 A shop gives its customer one coupon for every one dollar he has paid in one purchase (The cents are neglected). The number y of the coupons that a customer gets is a function of the money x he has paid. Express it.

Solution. If he has paid less than one dollar then he gets no coupon; i.e. $y=0$, $0\leqslant x<1$. If he has paid less than 2 dollars but not less than 1 dollar, then he gets 1 coupon; i.e. $y=1$, $1\leqslant x<2$; If he has paid less than 3 dollars but not less than 2 dollars, then he gets 2 coupons; i.e. $y=2$, $2\leqslant x<3$; etc. (Suppose that any payment is available). This funotion can be expressed in a more concise form:

$$y=\begin{cases}0, & 0\leqslant x<1\\ 1, & 1\leqslant x<2\\ 2, & 2\leqslant x<3\\ \cdots\cdots\cdots\cdots & \end{cases}$$

The graph of this function is the set of all those (x,y), such that y is the corresponding value of this function determined by x and x is any positive number. Thus, the graph of this coupon function is shown as follows.

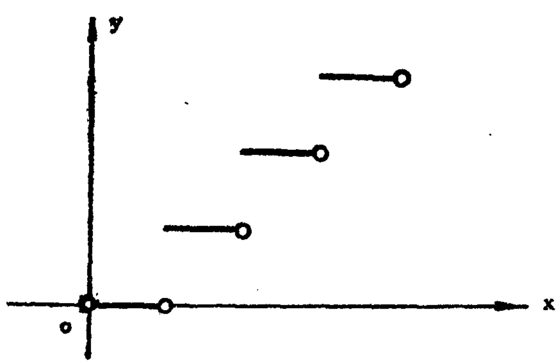


fig 1 5.6

From this interesting example we see that