GRADUATE STUDIES IN MATHEMATICS 185

Braid Foliations in Low-Dimensional Topology

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American Mathematical Society

This book is a self-contained introduction to braid foliation techniques, which is a theory developed to study knots, links and surfaces in general 3-manifolds and more specifically in contact 3-manifolds. With style and content accessible to beginning students interested in geometric topology, each chapter centers around a key theorem or theorems. The particular braid foliation techniques needed to prove these theorems are introduced in parallel, so that the reader has an immediate "take-home" for the techniques involved.

The reader will learn that braid foliations provide a flexible toolbox capable of proving classical results such as Markov's theorem for closed braids and the transverse Markov theorem for transverse links, as well as recent results such as the generalized Jones conjecture for closed braids and the Legendrian grid number conjecture for Legendrian links. Connections are also made between the Dehornoy ordering of the braid groups and braid foliations on surfaces.

All of this is accomplished with techniques for which only mild prerequisites are required, such as an introductory knowledge of knot theory and differential geometry. The visual flavor of the arguments contained in the book is supported by over 200 figures.

ISBN 978-1-4704-3660-5

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2010 *Mathematics Subject Classification*. Primary 57M25, 57R17, 57M50, 57M27, 20F36.

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Library of Congress Cataloging-in-Publication Data

LaFountain, Douglas J., 1978- | Menasco, William W., 1954-
Title: Braid foliations in low-dimensional topology / Douglas J. LaFountain, William W. Menasco.
Description: Providence, Rhode Island : American Mathematical Society, [2017] | Series: Graduate studies in mathematics ; volume 185 | Includes bibliographical references and index.
Identifiers: LCCN 2017016585 | ISBN 9781470436605 (alk. paper)
Subjects: LCSH: Braid theory–Textbooks. | Foliations (Mathematics)–Textbooks. | Topology–Textbooks. | AMS: Manifolds and cell complexes – Low-dimensional topology – Knots and links in S^3. msc | Manifolds and cell complexes – Differential topology – Symplectic and contact topology. msc | Manifolds and cell complexes – Low-dimensional topology – Geometric structures on low-dimensional manifolds. msc | Manifolds and cell complexes – Low-dimensional topology – Invariants of knots and 3-manifolds. msc | Group theory and generalization – Special aspects of infinite or finite groups – Braid groups; Artin groups. msc
Classification: LCC QA612.23 .L34 2017 | DDC 514/.72–dc23
LC record available at https://lccn.loc.gov/2017016585

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To Jessica and Melissa, and of course Joan

Preface

This book is an introduction to *braid foliation techniques*, which is a theory developed to study knots and links and related surfaces in 3-manifolds, and which from its inception has been intimately related to contact topology. The original idea of braid foliation techniques is due to Daniel Bennequin, who in the early 1980s first used a preliminary version of them to study transverse links and contact structures on $\mathbb{R}^3$, and established the existence of non-contactomorphic contact structures on $\mathbb{R}^3$. In the 1990s Joan Birman and William Menasco developed and systematized the theory of braid foliations, and in a series of papers spanning over ten years they used these techniques in $\mathbb{R}^3$ and S^3 to probe the landscape of closed braids representing topological link types, with their work culminating in the Markov Theorem without Stabilization and accompanying applications to the study of link types whose transverse classification is non-trivial. A number of researchers have since applied braid foliation techniques in new ways to solve foundational problems in braid theory and contact topology, most notably in Ivan Dynnikov and Maxim Prasolov's proof of the Legendrian grid number conjecture and the generalized Jones conjecture. Along the way, Tetsuya Ito and Malyutin-Netsvetaev discovered interesting interplay between braid foliations and Dehornoy's ordering of braids; furthermore, Ito and Keiko Kawamuro have recently extended the bulk of braid foliation techniques to arbitrary closed 3-manifolds supported by open book decompositions, terming this new generalized theory that of *open book foliations*.

We therefore believe that braid foliation techniques can be a highly useful implement in the toolbox of low-dimensional geometric topologists, and we have endeavored to present an accessible and detailed introduction to the theory in this book, including all of the above applications of braid foliation techniques. The primary audience that we have in mind is graduate students in geometric topology, but we hope that this work will also prove useful to the more experienced researcher as well. Rather than present all the details of braid foliation techniques at the outset, and overwhelm the reader with meaningless detail, we have constructed the book so that each chapter centers around a key theorem or collection of theorems, and the particular braid foliation techniques needed to prove that theorem are introduced in parallel, so that the reader has an immediate "take-home" for the techniques involved. The book does not need to be read entirely linearly, but we do recommend that the reader new to the subject read Chapters 2 and 3 in detail, as these two chapters form the core of braid foliation techniques. Following these chapters, the reader interested in transverse links in the standard contact structure may turn to Chapters 4, 5, 6 and 7; the reader interested in Legendrian links, including the work of Dynnikov and Prasolov and relations of braid foliations to convex surface theory, can skip ahead to Chapters 8, 9, and 12; those interested in studying braids algebraically can turn to Chapter 10; and the reader interested in Ito and Kawamuro's theory of open book foliations can proceed to Chapter 11.

An exercise section has been included at the end of each chapter, and we encourage the student to take time to work through these exercises before proceeding to the next chapter. Braid foliation techniques are highly visual, and we have therefore freely included figures throughout the book that will hopefully help the reader gain his or her own insight into the theory. We have also tried to point the reader to other powerful tools which can be used to solve similar problems to those addressed here, most notably, characteristic foliation and convex surface theory techniques of Emmanuel Giroux and Ko Honda, knot Floer homology theories of Peter Ozsváth and Zoltan Szabó, and knot contact homology techniques of Lenny Ng. In fact there are still open questions of how best to understand braid foliation techniques in the context of these other theories, which we encourage the interested reader and researcher to pursue.

One final comment: We emphasize that throughout the book our links will be oriented and smooth, and our ambient isotopies will be smooth. This will allow us to have well-defined notions of transversality, and utilize standard applications of Sard's theorem involving regular projections and general position. At times, however, we will find it very useful to employ various piecewise-linear approximations of our smooth links to organize combinatorial arguments. At the outset, therefore, we note that these piecewise-linear approximations will always have a smooth link close by, to which we are actually performing smooth isotopies.

Acknowledgements. The authors are indebted to the input and helpful comments of a number of scholars and students. In particular, we would like to thank John Etnyre, Keiko Kawamuro and Tetsuya Ito for their close reading of, and comments on, early versions of our book. Additionally we thank the anonymous AMS referees for numerous comments that made our expository presentation clearer and more accessible. Doug LaFountain thanks the Western Illinois University Foundation and the Office of Sponsored Projects for the summer grant which enabled his work on the book. William Menasco thanks his 2016 Spring semester MTH828 Topology Topics graduate class for their patient close reading of early versions of our book, and their many helpful comments. Finally, we thank Matias Dahl for his permission to use three figures of contact structures which he created using the CSPlotter package developed by him in Mathematica; these are Figures 2, 3 and 8 in Chapter 4.

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Contents

Chapter 1

Links and closed braids

The primary low-dimensional topological objects of interest throughout this book will be oriented 1-, 2- and 3-manifolds and specific topological (and at times geometric) relationships amongst them. In this first chapter we begin with closed oriented 1-manifolds embedded in the 3-sphere and we establish what it means to study these objects as closed braids. The central theorem for this chapter will be Alexander's theorem [**Ale23**], which shows that every link can be braided via an isotopy; we will present two proofs of this theorem, as well as discuss a number of useful ways to visualize and organize links and braids.

1.1. Links

An oriented link of m components in S^3 is a smooth embedding of m disjoint oriented circles into the 3-sphere, and two links L_0 and L_1 are said to be in the same link type $\mathcal{L}$ if there is a smooth isotopy of embedded links L_t ($t \in [0,1]$) connecting them. It is standard to view S^3 as the one-point compactification of $\mathbb{R}^3$ and to alternatively think of this isotopy L_t, and the accompanying links, as being in S^3 or $\mathbb{R}^3$; see for example [**Rol90**]. We will adopt this approach throughout this book.

An important theorem, which we will state without proof here (see [**Rei74**] for example) is the following:

Theorem 1.1. *Two links L_0 and L_1 are isotopic in $\mathbb{R}^3$ via an isotopy L_t for $t \in [0,1]$ if and only if there is an ambient smooth isotopy $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ such that $L_t = \phi_t(L_0)$ for all $t \in [0,1]$.*

Links in S^3, and for that matter links in general 3-manifolds, stand as important building blocks in low-dimensional topology: for example, any

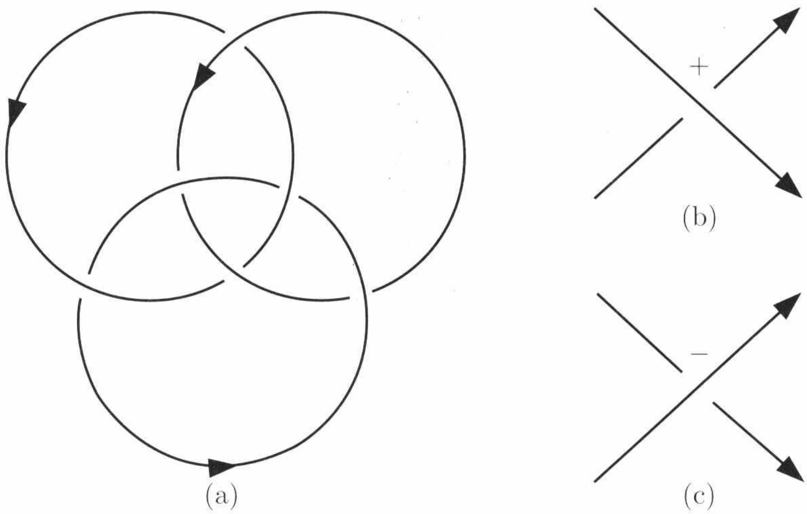

Figure 1. A regular projection of the Borromean rings, and examples of a positive and negative crossing.

closed oriented 3-manifold can be constructed via surgery on a link in S^3 [**Lic97**], and since S^3 is the boundary of the 4-ball B^4, links can also be used to construct 4-manifolds [**Kir89**]; the complements of links can yield important examples of hyperbolic 3-manifolds [**Thu82**]; links can bound embedded surfaces which provide interesting fibrations of S^3 [**Rol90**]; and links provide an important way to construct quantum field theories in high-energy physics [**Ati90**].

Since all of the above constructions will yield identical results for two isotopic links, one basic question concerning links is determining whether two links L_1 and L_2 in the same 3-manifold are isotopic or not. For this purpose, a panoply of *invariants* of links has been identified, both classical and modern, which provides numerous possible ways to distinguish between two links L_1 and L_2 which represent different link types $\mathcal{L}_1$ and $\mathcal{L}_2$. Such invariants assign some quantity (either a number, polynomial, group, graded ring, vector space, etc.) to any given link, such that this assignment does not vary when an isotopy of that link occurs. Examples include the genus of a link, the unknotting number, minimal braid index, the fundamental group of the complement of a link, various polynomial invariants such as the Alexander polynomial, Jones polynomial and HOMFLYPT polynomial, and categorifications of these in knot Floer homology and Khovanov homology.

Our focus in this book will not be to distinguish between links in distinct isotopy classes, but rather to investigate the internal structure within a fixed link type. As such, our primary goal will not be to work with the above invariants. However, the interested reader who is familiar with the above