# MATHEMATICAL PROBLEMS of RELATIVISTIC PHYSICS

# Mathematical Problems of Relativistic Physics

*by*

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WITH AN APPENDIX ON

Group Representations in Hilbert Space

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1963

AMERICAN MATHEMATICAL SOCIETY

PROVIDENCE, RHODE ISLAND

The Summer Seminar was conducted, and the proceedings prepared in part, by the American Mathematical Society under the following contracts and grants:

Grant NSF-G12432 from the National Science Foundation.

Contract No. AT(30-1)-2482 with the United States Atomic Energy Commission.

Contract Nonr-3081(00) with the Office of Naval Research.

Contract DA-19-020-ORD-5086 with the Office of Ordnance Research.



Library of Congress Catalog Card Number 62-21480

*Printed in the United States of America*

## Foreword

This is the second of a series of four volumes which are to contain the Proceedings of the Summer Seminar on Applied Mathematics, arranged by the American Mathematical Society and held at the University of Colorado for the period July 24 through August 19, 1960. The Seminar was under the sponsorship of the National Science Foundation, Office of Naval Research, Atomic Energy Commission, and the Office of Ordnance Research.

For many years there was an increasing barrier between mathematics and modern physics. The separation of these two fields was regrettable from the point of view of each—physical theories were largely isolated from the newer advances in mathematics, and mathematics itself lacked contact with one of the most stimulating intellectual developments of our times. During recent years, however, mathematicians and physicists have displayed alacrity for mutual exchange. This Seminar was designed to enlarge the much-needed contact which has begun to develop.

The purpose of the Seminar was primarily instructional, with emphasis on basic courses in classical quantum theory, quantum theory of fields and elementary particles, and statistical physics, supplemented by lectures specially planned to complement them. The publication of these volumes is intended to extend the same information presented at the Seminar to a much wider public than was privileged to actually attend, while at the same time serving as a permanent reference for those who did attend.

Following are members of a committee who organized the program of the Seminar:

Kurt O. Friedrichs, Chairman
Mark Kac
Menahem M. Schiffer
George E. Uhlenbeck
Eugene P. Wigner

Local arrangements, including the social and recreational program, were organized by a committee from the University of Colorado, as follows:

Charles A. Hutchinson
Robert W. Ellingwood

The enduring vitality and enthusiasm of the chairmen, and the cooperation of other members of the university staff, made the stay of the participants extremely pleasant; and the four agencies which supplied financial support, as acknowledged on the copyright page, together with the Admissions Committee, consisting of Bernard Friedman, Wilfred Kaplan, and Kurt O. Friedrichs, Chairman, also contributed immeasurably to the successful execution of the plans for the Seminar.

The Seminar opened with an address given by Professor Mark Kac, Department of Mathematics, Cornell University, on the subject "A Mathematician's Look at Physics: What Sets us Apart and What May Bring us Together." Afternoons were purposely kept free to give participants a chance to engage in informal seminars and discussions among themselves and with the distinguished speakers on the program.

*Editorial Committee*

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# Preface

This book gives the approximate text of a course of eight lectures from combined rigorous mathematical and physically conceptual viewpoints, supplemented by two more purely mathematical lectures. The main purpose is to provide an up-to-date introduction, for the mathematically trained reader, to the central mathematical features of fundamental relativistic physics. While we have aimed for accuracy and scope of perspective rather than for completeness of detail, this purpose itself seemed better served by the inclusion of several detailed discussions and the omission of any significant treatment of many important topics, whose inclusion would not in our judgement have altered the essential form which we have attempted to delineate. In particular, the theory is very largely presented in terms of Bose-Einstein quantum fields, Fermi-Dirac fields being brought in only very briefly and in a descriptive way.

A relatively informal lecture style seemed the best adapted to the quite challenging task of formulating the mathematically intelligible essence of such a complex and sophisticated subject as quantum field and particle theory with the requisite conciseness. No attempt has been made to change this form of presentation in the printed text, in view of its apparent appropriateness for this task.

While the mathematical beauty and inevitability of many parts of modern relativistic physics are now clearly visible, there remain unresolved foundational questions, which in fact dominate the scientific area being considered. It is our conviction that quantum field theory, at least, is on the verge of becoming mathematically firmly established, and will in fact in a few years be recognized as closely parallel to the analytical theory of functionals over infinite-dimensional non-linear manifolds admitting group-invariant differential-geometric structures. In any event, we hope to have given some measure of the recent advances in the subject, and to have conveyed some feeling for the magnificent intuitive scientific structure which has yet to be fully understood mathematically.

Special thanks are due Leonard Gross and David Shale for scientifically useful comments, as well as to the former for help with the original notes.

# Varieties of Approaches

To clarify our general intentions and purposes in these chapters, let us review very briefly the varieties of approaches to quantum fields and particles which are currently popular. Although the *ultimate* aims of many theoreticians are rather similar, involving mainly an increase in our understanding of fundamental physical phenomena, their shorter-term objectives are quite varied, so much so that fundamental theoretical physics has a rather fragmented appearance at present.

The traditional approach effectively regarded theoretical physics as a game whose purpose was to derive from simple theoretical principles the abstruse numbers obtained in laboratory experiments on particles; this description is a variant of one due to Dirac. The great success of Dirac, Heisenberg, Schrödinger, and many others at this game during the late twenties laid incidentally the foundations of modern quantum theory. But in the past thirty years the game has proved so difficult that people have generally felt forced to modify its rules in one way or another.

The success of the renormalization theory initiated in clear-cut form chiefly by Feynman, Schwinger, and Tomonaga, in computing with great accuracy quantum radiation effects on the electron, represents the most remarkable theoretical explication of fundamental physical data in the past thirty years. It was based however on a certain relaxation of the rules permitting the use of an ad hoc argument at a crucial stage in the computation to resolve a serious difficulty, i.e. eliminate the so-called divergences to which the theory and mathematical procedure led. This remains the case today despite the considerable simplifications and clarifications due to Dyson, Ward, Salam, van Hove and his associates, and many others.

More recently the "axiomatic" schools which have emerged from this situation have surpassed the traditional approach in logical clarity, utilizing an explicit rather than implicit statement of their fundamental principles. They have concentrated on increasing understanding of the meaning, scope, and general implications of quantum field theory, and have effectively given up the attempt to compute experimental data from theoretical principles. The most active of these "schools," including notably that of Källén-Wightman and that of Lehmann-Symanzik-Zimmermann (to both of which Haag and Jost have made significant contributions), are rather mathematical in spirit but do not always distinguish between mathematically rigorous and partially heuristic definitions and results. From an overall point of view, however, the main problem here is the lack as yet of non-trivial examples of systems satisfying the axioms, i.e. systems involving real emission and absorption of particles.

Roughly at the other end of the theoretical spectrum from the axiomatic schools are those concerned chiefly with the correlation of experimental data by means of approximations to and heuristic techniques in quantum field theory of varying degrees of physical motivation and, unfortunately, quite uncertain reliability. In any event, the ideas of Chew, Goldberger, and Low have proved to be particularly useful in reducing the large and rapidly growing volume of experimental data in nuclear physics. The technique of so-called "dispersion relations" has been widely used for a substantial time, and some of the relations have been supported by experimental evidence, but a clear-cut formulation and derivation of the relations within a rigorous mathematical framework has not yet been given, and it also seems quite difficult in the nature of things to make a conclusive experimental test of the relations, since, unlike the familiar relations that have been so tested, the checking of an individual numerical equality in a dispersion relation necessarily involves measurements at all, including arbitrarily high, energies.

These three schools have certain connections, a particularly interesting and actively investigated one being that between the empirically-oriented and the axiomatic schools via the theory of dispersion relations. But on the whole there does not appear to be much prospect for their fundamental unification in the foreseeable future. On the other hand, until the elementary question, of what, precisely, a quantum field theory consists of, is answered in satisfactory physical and mathematical terms, there are insufficient rational grounds for pessimism or optimism.

A pure mathematician who is interested in fundamental physics will see at once that there is another possible approach, that of building up on the bedrock of rigorous mathematics, while keeping as close as possible to the ideas that emerge from empirical practice. Ten years ago such an approach might have seemed very naïve, but by now it is clear that the rigor and mathematical method, far from proving burdensome, enable one to deal simply and definitely, if in a rather sophisticated way, with some of the really significant theoretical questions; and that the close connection between that which is mathematically viable and physically meaningful is a rather general feature of the situation, and not limited to such cases as that treated by Bohr and Rosenfeld in their classical work on the measurability of the electromagnetic field.

Our purpose here is mainly to treat those parts of the theory of fields and particles which are now available in a rigorous, compact, and general form. The solution of the relevant problems has tended to lead to new problems, some of which we shall describe. We shall have to pay the price of increasing at least temporarily the difficulty of making a dictionary for translating between experimental physics and mathematics. We must not expect too much direct physical contact too soon, in view of the very substantial complications inherent in any comprehensive theory conceivably applicable to elementary particle interactions. But the pursuit of this game of capturing modern physical ideas and principles in rigorous and simple mathematics is a reasonable and interesting activity in itself. We think moreover that there are now visible lines of development offering definite promise of dealing effectively with physically interesting relativistic interactions.

From a purely mathematical point of view the main mathematical fields pertinent to the general theory of particles and fields are:

1. Operator theory (especially operator algebras).
2. Theory of group representations (especially of the Lorentz and other physical symmetry groups).
3. Theory of functionals.
4. Theory of partial differential equations.

Large parts of these subjects are relevant here, in fact a year's course on each of them would not be amiss. Of course, here we can treat only a few aspects of special relevance. We shall say only a little about operator theory, and less about group representations, as these will be treated in Professor Mackey's chapters. We shall discuss analysis in function space, because of its relevance and relative novelty, and note its relation to the line of development originating with the work of Wiener on Brownian motion. We shall do little with the theory of partial differential equations, partly because the aspects of the theory of greatest relevance—the global spectral theory of variable coefficient and non-linear hyperbolic equations—are as yet rather undeveloped.

# Contents

CHAPTER I

# Quantum Phenomenology

We begin by treating the notion of a physical system, keeping as closely as possible to the use of concepts having a fairly direct empirical or physically intuitive significance. The fundamental object associated with a physical system may be taken either as an *observable* or as a *state*. The former concept seems simpler from a naïve point of view, and leads to a viable theory in terms of which state may be treated quite effectively, so we shall start with observable as a fundamental undefined notion.

We should mention parenthetically that the early formulation of quantum phenomenology asserted that: (1) an observable is a self-adjoint operator in a Hilbert space; (2) a state is a vector $\psi$ in this space; the connection between (1) and (2) being that the expectation value of $A$ in the state $\psi$ is $(A\psi, \psi)$. These "axioms" are technically simple, but they are thoroughly unintuitive and ad hoc. In addition, it has turned out recently that they are technically really effective only in the case of systems of a finite number of degrees of freedom. In fact certain of the ultraviolet divergences of quantum field theory result indirectly from the inadequacy of the older phenomenology. Therefore there is ample reason, both foundational and technical, to prefer the more recent form, which is given below.

Now both physically and mathematically it appears that the bounded observables play the fundamental role, the unbounded ones being readily dealt with in terms of the bounded ones, as far as foundational purposes are concerned. Taking e.g. a one-dimensional quantum-mechanical particle, no given finite physical apparatus can conceivably accurately measure the momentum $p$, once this momentum goes beyond a certain limit. Now one may construct larger and more refined apparatus, and thereby for each finite $n$, measure $F_n(p)$, where $F_n(x) = x$ for $|x| \leqq n$ and $F_n(x) =$ say $n \operatorname{sgn} n$ for $|x| > n$. That is, one can measure the infinite sequence of observables $F_1(p), F_2(p), \cdots$, each of which is bounded; and $p$ itself is not measurable directly but only as a limit of such a sequence, and so involves an unphysical infinity of experiments. On the other hand, mathematically we are entitled to look ahead a bit, to the first axioms of early quantum phenomenology, to the effect that an observable is a self-adjoint operator in a Hilbert space. The well-known great difficulty of performing effectively simple algebraic operations on non-commuting self-adjoint operators together with the various possibilities for treating unbounded in terms of bounded operators, strongly suggest the limitation to the bounded ones.

So we consider to begin with only the bounded observables of the given physical system. From an intuitive point of view it is clear that if $A$ is a bounded observable and $\alpha$ is a real number, then $\alpha A$ is a bounded observable; it is measured simply by measuring $A$ and multiplying the result by $\alpha$. Similarly $A^2$ is a bounded observable, measured by measuring $A$ and squaring the result. Now if $B$ is another observable, the sum $A + B$ and product $AB$ can be similarly defined only when $A$ and $B$ are simultaneously observable. We may however define $A + B$ in a more indirect physical fashion as that observable whose expectation in any state is the sum of the expectations of $A$ and of $B$. Intuitively it is plausible that an observable may be reconstructed from its expectation values in all states; alternatively this definition may be regarded as a restriction on the states of the system. On the other hand, the product $AB$ may not be defined in a similar fashion because it is not even true for simultaneously observable $A$ and $B$ that the expectation value of the product is the product of the expectation values (as is familiar in the theory of probability, whose observables are usually called "random variables").

Thus it is physically reasonable to postulate that the bounded observables of the physical system form a type of algebra, the relevant operations being multiplication by scalars, squaring and addition of observables, but not multiplication in general. However, in view of the indirect character of the definition of addition, the full reasonableness of the assumption that two observables can be added will follow only if the theory which is built up from such assumptions has as a logical consequence the rationalizing assumptions that the expectation value of the sum of two observables is the sum of their expectation values, in a particular state, and that any observable can be recovered from its expectation values.

In addition it is reasonable to assume that there is a unit observable $I$ whose expectation value in every state is unity, and that the usual rules for the reduction of measurements of simultaneously measurable observables are valid. This last requirement turns out to be needed only in the form

$$A^r \circ A^s = A^{r+s}, \qquad (\alpha A)^r = \alpha^r A^r$$

if the pseudo-product $A \circ B$ is defined by the equation

$$A \circ B = \frac{1}{4}[(A + B)^2 - (A - B)^2],$$

and $A^r$ is defined recursively by the equations

$$A^0 = I, \qquad A^r = A \circ A^{r-1}.$$

Since $A \circ B$ coincides with the phenomenological product described above when $A$ and $B$ are simultaneously observable, the present requirements have immediate intuitive validations.

Thus we have rationalized the following mathematical axiom:

PHENOMENOLOGICAL POSTULATE, ALGEBRAIC PART: *A physical system is a collection of objects, called (bounded) observables, for which operations of multiplication by a real number, squaring, and addition are defined, and satisfy the usual assumptions for a linear vector space as well as those involving the squaring operation given above.*

As a mathematical example, consider the set of all bounded hermitian (linear, everywhere defined) operators on a Hilbert space. It is obvious that with the usual algebraic operations the foregoing postulate is satisfied. It may be helpful to note incidentally that the conventional product of operators is not meaningful within this system, since the product of two hermitian operators will again be hermitian only when they commute; while the pseudo-product $A \circ B = (AB + BA)/2$ in the present case has for non-commuting hermitian $A$ and $B$ no physical interpretation.

Now the main result we need in the present connection is the

BASIC PHENOMENOLOGICAL PRINCIPLE: *Any physical system is determined in all its physically observable aspects by its algebra of bounded observables.*

That is to say, two systems whose bounded observables may be brought into one-to-one correspondence, in such a fashion that sums, squares, and products by real numbers correspond, are physically identical—apart from the labelling of the observables.

To explain more precisely what is meant by a "physically observable aspect," let us introduce the key notions of state, pure state, and spectral (exact possible) value of an observable. From an empirical standpoint, a state $E$ exists only as a rule which assigns to each bounded observable its expectation value in the state; any possible metaphysical distinction between the state and the corresponding functional on the observables is irrelevant for empirical objectives. Accordingly, we define a state as this functional and the following properties of a state $E$ have a clear intuitional validity:

1. *Linearity*: $E(A + B) = E(A) + E(B)$,
$$E(\alpha A) = \alpha\, E(A),$$
if $A$ and $B$ are any bounded observables and $\alpha$ is a real number.

2. *Positivity*: $E(A^2) \geqq 0$.

3. *Normalization*: $E(I) = 1$.

Thus on a rather conservative physical basis a state must be some sort of normalized positive linear functional on the observables. For basic phenomenological purposes this is all that turns out to be required for a state, and so we follow von Neumann in *defining* a state as such a functional.

Now if a system is in a state $E$ with probability $\alpha$ and in a state $E'$ with probability $\alpha'$, where $\alpha + \alpha' = 1$ and $\alpha > 0$, $\alpha' > 0$, the effective state of the system is $E''$, where

$$E''(A) = \alpha E(A) + \alpha' E'(A).$$

The state $E''$ is called a *mixture* of the states $E$ and $E'$ and following Weyl we call a state *pure* if it cannot be represented as a mixture of two distinct states. It is evident that it is the pure state that plays the fundamental part in non-statistical mechanics; an experiment of maximal theoretical accuracy will yield a pure state of the system.

To clarify these notions, consider briefly the system of all bounded hermitian operators in a Hilbert space $\mathscr{H}$. If $\psi$ is any unit vector in $\mathscr{H}$, the functional $E$ defined by the equation

$$E(A) = (A\psi, \psi)$$

is easily seen to be a state. It is actually a pure state, as can be seen in a fashion that will be indicated later. In many conventional treatments of quantum mechanics, the vector $\psi$ is called a state, but it is evidently of quite another character from the functional $E$ which is here defined as a state. In particular $\psi$ is incompletely physically observable, any multiple of $\psi$ by a number of unit modulus being physically indistinguishable from it. Here $\psi$ will be referred to as a *state vector* or *wave function* for the state $E$. Incidentally, it is only in the trivial case of the finite-dimensional Hilbert space $\mathscr{H}$ that every pure state has the foregoing form; for an infinite-dimensional space there are others, which can arise, e.g., from the continuous spectrum which may manifest itself for operators in an infinite-dimensional space.

An example of a mixed state is provided by one of the form

$$E(A) = \operatorname{tr}(AD),$$

where $D$ is a non-negative self-adjoint operator of absolutely convergent trace, and total trace unity. $D$ is then uniquely determined by $E$ and is called the "von Neumann density operator (matrix)." Such a state is pure if and only if $D$ is of unit rank, in which case it arises from a wave function in the fashion just indicated.

To arrive at the notion of spectral value, the variance of an observable $A$ in a state $E$ may be reasonably defined as the quantity $E(A^2) - E(A)^2$, which is automatically non-negative by virtue of the positivity of the functional $E$. In line with this, $A$ may be said to *have an exact value in the state* $E$ in case its variance vanishes; and the values $E(A)$ of $A$ in all such states designated as the spectrum of the observable $A$.

Now it is clear from the definitions of state, pure state, and spectral value, that they are wholly determined by the algebra of bounded observables. But this is significant only if states and pure states exist in ample number, and if spectral values exist, and relate to states in the usual probabilistic fashion (i.e. the expectation of the observables is the average of the spectral values with respect to a probability distribution determined by the state), etc. To prove such results the phenomenological postulate above must be supplemented by a postulate making possible the application of analytical methods.

To arrive at a physically meaningful postulate that will be mathematically effective, consider the properties which may be anticipated for the bounds of the observables, whose finiteness has not thus far been utilized. The bound represents, in an intuitive physical way, the greatest possible absolute value for the observable. This interpretation together with a quite moderate amount of reflection shows the physical basis for the

PHENOMENOLOGICAL POSTULATE, ANALYTICAL PART: *To each observable $A$ is assigned a "bound," designated $\|A\|$, in such a way that the following conditions are satisfied:*

i. $\|A\| \geqq 0$ *and* $\|A\| = 0$ *if and only if* $A = 0$.

ii. $\|\alpha A\| = |\alpha|\,\|A\|$ *and* $\|A + B\| \leqq \|A\| + \|B\|$.

iii. *The collection of all observables is complete with respect to the metric determined by the bound, i.e. if $A_1, A_2, \cdots$ is a sequence of observables such that $\|A_m - A_n\| \to 0$ as $m, n \to \infty$, then there exists an observable $A$ such that $\|A_n - A\| \to 0$.*

iv. $\|A^2\| = \|A\|^2$ *and* $\|A^2 - B^2\| \leqq \text{Max}\,[\|A^2\|, \|B^2\|]$.

v. $A^2$ *is a continuous function of* $A$, *i.e. if* $A_n \to A$, *then* $A_n^2 \to A^2$.

Conditions i and ii have a direct physical justification. The condition iii is virtually a matter of convenience, for an incomplete system could always be completed; an observable $A$ could be defined if necessary, its expectation value in a state being explicitly obtainable as the limits of the expectation values of the $A_n$. Condition iv takes a slight amount of reflection for its intuitive justification. Condition v merely asserts that if two observables are close (as measured by the bound of their difference) then so are their squares.

For an example, consider again the system of all bounded hermitian operators on a Hilbert space, with $\|A\|$ defined as the usual bound of the operator $A$. That is, $\|A\|$ is the least upper bound of the Hilbert space norms $\|A\psi\|$ as $\psi$ varies over all unit vectors, or equivalently, for hermitian operators, of $|(A\psi, \psi)|$. All of the foregoing conditions follow almost trivially.

On the strength of the combined algebraic and analytical parts of the phenomenological postulates, all of the physically plausible and conventionally accepted principles of quantum phenomenology may be rigorously established. The proofs are based on now familiar results and methods of abstract analysis, including notably the Stone-Gelfand representation theory and such results in linear analysis as the Hahn-Banach, Krein-Milman, and Riesz-Markoff theorems.

Among the results are:

1. *There exists an ample supply of pure states, in the sense that two observables having the same expectation values in all pure states must be identical.* In particular, the justification for the assumption that two observables can be added is completed.

2. *Any observable admits a closed set of spectral values, and the expectation of the observable in any state is the average of these spectral values with respect to a probability distribution on them canonically determined by the state.* Specifically, this distribution may be defined as that with characteristic function $E(e^{itA})$, where $E$ is the state and $A$ the observable (here $e^{itA}$ is defined in the obvious fashion, or alternatively, $E(e^{itA})$ may be replaced by $E(\cos tA) + iE(\sin tA)$, where $\cos tA$ and $\sin tA$ are defined by the demonstrably convergent, conventional power series expansions). It is not difficult to see that this function (of $t$) is positive definite and the Fourier-Stieltjes transform of a probability distribution.

3. *The smallest closed system of observables (in the sense of the*