INTERDISCIPLINARY APPLIED MATHEMATICS

MECHANICS AND MATERIALS

Computational Inelasticity

J.C. Simo
T.J.R. Hughes

计算非弹性

Springer

世界图书出版公司
www.wpcbj.com.cn

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图书在版编目（CIP）数据

计算非弹性=Computational Inelasticity：英文／（美）思摩（Simo,J.C.）著. —北京：世界图书出版公司北京公司，2008.5
ISBN 978-7-5062-9225-2

Ⅰ.计… Ⅱ.思… Ⅲ.弹性力学-英文 Ⅳ.0343

中国版本图书馆CIP数据核字（2008）第055591号

书　　名：Computational Inelasticity
作　　者：J. C. Simo, T. J. R. Hughes
中 译 名：计算非弹性
责任编辑：高蓉 刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：010-64015659
电子信箱：kjsk@vip.sina.com

开　　本：24开
印　　张：17
版　　次：2008 年 05月第 1 次印刷
版权登记：图字:01-2008-2087
书　　号：978-7-5062-9225-2 / O・616　　定　　价：59.00 元

J.C. Simo (deceased)
Formerly, Professor of Mechanical Engineering
Stanford University
Stanford, CA 94305
USA

T.J.R. Hughes
Mechanics and Computation
Durand Building
Stanford University
Stanford, CA 94305
USA

Editors

J.E. Marsden
Control and Dynamical Systems
107-81
California Institute of Technology
Pasadena, CA 91125
USA

L. Sirovich
Division of Applied Mathematics
Brown University
Providence, RI 02912
USA

S. Wiggins
Control and Dynamical Systems
107-81
California Institute of Technology
Pasadena, CA 91125
USA

Mathematics Subject Classification (1991): 73EXX, 73FXX, 65MXX, 73CXX

Library of Congress Cataloging-in-Publication Data
Simo, J.C. (Juan C.), 1952–1994
Computational inelasticity / J.C. Simo, T.J.R. Hughes.
p. cm. — (Interdisciplinary applied mathematics ; 7)
Includes bibliographical references and index.
ISBN-13: 978-0-387-97520-7 (hardcover : alk. paper)
1. Elasticity. 2. Viscoelasticity. I. Hughes, Thomas J.R.
II. Title. III. Series: Interdisciplinary applied mathematics ; v. 7.
QA931.S576 1997
531'.382—dc21 97-26427

9 8 7 6 5 4 3 (Corrected second printing, 2000)

ISBN-13: 978-0-387-97520-7

springer.com

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Preface

This book goes back a long way. There is a tradition of research and teaching in inelasticity at Stanford that goes back at least to Wilhelm Flügge and Erastus Lee. I joined the faculty in 1980, and shortly thereafter the Chairman of the Applied Mechanics Division, George Herrmann, asked me to present a course in plasticity. I decided to develop a new two-quarter sequence entitled "Theoretical and Computational Plasticity" which combined the basic theory I had learned as a graduate student at the University of California at Berkeley from David Bogy, James Kelly, Jacob Lubliner, and Paul Naghdi with new computational techniques from the finite-element literature and my personal research. I taught the course a couple of times and developed a set of notes that I passed on to Juan Simo when he joined the faculty in 1985. I was Chairman at that time and I asked Juan to further develop the course into a full year covering inelasticity from a more comprehensive perspective. Juan embarked on this path creating what was to become his signature course. He eventually renamed it "Computational and Theoretical Inelasticity" and it covered much of the material that was the basis of his research in material modeling and simulation for which he achieved international recognition. At the outset we decided to write a book that would cover the material in the course. The first draft was written quite expeditiously, and versions of it have been circulated privately among friends, colleagues, and interested members of the research community since 1986. Thereafter progress was intermittent and slow. Some things were changed and some new chapters were added, but we both had become distracted by other activities in the early 1990s. Prior to that, we frequently discussed what would be necessary "to get it out the door," but I do not recall the subject even coming up once in the years immediately preceding Juan's death in 1994. Since that time I have been repeatedly urged to bring the project to completion. Through the efforts of a number of individuals, the task is now completed.

This book describes the theoretical foundations of inelasticity, its numerical formulation, and implementation. It is felt that the subject matter described herein constitutes a representative sample of state-of-the-art methodology currently used in inelastic calculations. On the other hand, no attempt has been made to present a careful account of the historical developments of the subject or to examine in detail important physical aspects underlying inelastic flow in solids. Likewise, the list of references should, by no means, be regarded as a complete literature survey of the field.

Chapter 1 begins with an overview of small deformation plasticity and viscoplasticity in a one-dimensional setting. Notions introduced in Chapter 1 are generalized to multiple dimensions and developed more comprehensively in subsequent chapters. Ideas of convex optimization theory, which are the foundations of the numerical implementation of plasticity, are first introduced in Chapter 1. In Chapter 2 the theory is generalized to multiple dimensions. In addition to the three-dimensional case, plane-strain and plane-stress cases are presented, as well as thermodynamic considerations and the principle of maximal plastic dissipation. Chapter 3 deals with integration algorithms for the constitutive equations of plasticity and viscoplasticity. The two most important classes of return-mapping algorithms are described, namely, the closest-point projection and cutting-plane algorithms. The classical radial return method is also presented. Another important mathematical tool in the construction of numerical methods for inelastic constitutive equations, the operator-splitting methodology, is also introduced in Chapter 3. Chapter 4 deals with the variational setting of boundary-value problems and discretization by finite element methods. Key technologies for successful implementation of inelasticity, such as the assumed strain method and the B-bar approach, are described. The generalization of the theory to nonsmooth yield surfaces is considered in Chapter 5. Mathematical numerical analysis issues of general return-mapping algorithms and, in particular, their nonlinear stability are presented in Chapter 6. The generalization to finite-strain inelasticity theory commences in Chapter 7 with an introduction to nonlinear continuum mechanics, the notion of objectivity, variational formulations of the large-deformation case, and hyperelastic and hypoelastic constitutive equations. The practically important subject of objective integrative algorithms for rate constitutive equations is described in Chapter 8. In Chapter 9 the theory of hyperelastic-based plasticity models is presented. This chapter covers the local multiplicative decomposition of the deformation gradient into elastic and plastic parts and numerical formulations of this concept by way of return-mapping algorithms. Chapter 10 deals with small and large deformation viscoelasticity.

I believe a good, basic course of a semester's or quarter's duration would focus on Chapter 1 to 4. For more advanced students wishing to understand the large deformation theory, Chapters 7 and 8 are essential. Chapter 8, in particular, deals with the types of formulations commonly used in large-scale commercial computer programs. There is more research interest in the hyperelastic-based theories of Chapter 9, which are more satisfying from a theoretical point of view. However, as of this writing, they have not enjoyed similar attention from the developers of most commercial computer programs.

Over the past two years, this text has been used as the basis of courses at Stanford and Berkeley which provided vehicles for readying the manuscript for publication. I wish to sincerely thank the students in these classes for their considerable patience and effort. Present and past graduate students of Juan's and mine were also instrumental in bringing the endeavor to fruition. Among them I wish to thank, in

particular, Francisco Armero, Krishnakumar Garikipati, Sanjay Govindjee, John Kennedy, and Steve Rifai. However, without the hard work and devotion of two recent students, I doubt that this project would have been completed: Vinay Rao and Eva Petöcz critically read the manuscript and interacted with the other individuals who provided corrections. Vinay and Eva synthesized the inputs, made changes, and managed the master file containing the manuscript. They searched for and found lost drawings, and when missing figures could not be located, they drew them themselves. They spent many hours in this effort, and I wish to express my sincere thanks and gratitude to them.

Thomas J. R. Hughes
Stanford, March 1998

Contents

1

Motivation. One-Dimensional Plasticity and Viscoplasticity

In this chapter we consider the formulation and numerical implementation of one-dimensional plasticity and viscoplasticity models. Our objective is to motivate our subsequent developments of the theory in the simplest possible context afforded by a one-dimensional model problem. Since the main thrust of this monograph is the numerical analysis and implementation of classical plasticity, viscoplasticity, and viscoelasticity models, an attempt is made to formulate the basic governing equations in a concise form suitable for our subsequent numerical analysis. To this end, once a particular model is discussed, the basic governing equations are summarized in a BOX that highlights the essential mathematical aspects of the theory. Likewise, the corresponding numerical algorithms are also summarized in a BOX that highlights the essential steps involved in the actual numerical implementation. We follow this practice throughout the remaining chapters of this monograph.

1.1 Overview

An outline of the topics covered in this introductory chapter is as follows.

In Section 1.2 we present a detailed formulation of the governing equations for a one-dimensional mechanical device consisting of a linear spring and a Coulomb friction device. This simple model problem exhibits all the basic features underlying classical rate-independent (perfect) plasticity, in particular, the notion of *irreversible response* and its mathematical modeling through the Kuhn–Tucker complementarity conditions. Subsequently, we generalize this model problem to account for hardening effects and discuss the mathematical structure of two classical phenomenological hardening models known as *isotropic* and *kinematic* hardening.

In Section 1.3 we summarize the equations of the one-dimensional elastoplastic boundary-value problem and discuss the weak or variational formulation of these equations. Then we provide an outline of the basic steps involved in a numerical solution procedure. With this motivation at hand, in Section 1.4 we discuss the numerical integration of the constitutive models developed in Section 1.2

and introduce the fundamental concept of *return mapping* or *catching up* algorithm. As shown in Chapter 3 this notion has a straightforward generalization to three-dimensional models and constitutes the single most important concept in computational plasticity. In Section 1.5 we illustrate the role of these integrative algorithms by considering the simplest finite-element formulation of the elastoplastic boundary-value problem. We discuss the incremental form of this problem and introduce the important notion of *consistent* or *algorithmic tangent modulus*.

Finally, Section 1.6 generalizes the preceding ideas to accommodate rate-dependent response within the framework of classical viscoplasticity. We examine two possible formulations of this class of models and discuss their numerical implementation. In particular, emphasis is placed on the significance of viscoplasticity as a *regularization* of rate-independent plasticity. This interpretation is important in the solution of boundary-value problems where hyperbolicity of the equations in the presence of softening can always be attained by suitable choice of the relaxation time.

For further reading on the physical background, and generalizations, see Lemaitre and Chaboche [1990].

1.2 Motivation. One-Dimensional Frictional Models

To motivate the mathematical structure of classical rate-independent plasticity, developed in subsequent sections, we examine the mechanical response of the one-dimensional frictional device illustrated in Figure 1.1.

We assume that the device initially possesses unit length (and unit area) and consists of a spring, with elastic constant E, and a Coulomb friction element, with constant $\sigma_Y > 0$, arranged as shown in Figure 1.1. We let σ be the applied stress (force) and ε the total strain (change in length) in the device.

1.2.1 Local Governing Equations

Inspection of Figure 1.1 leads immediately to the following observations:

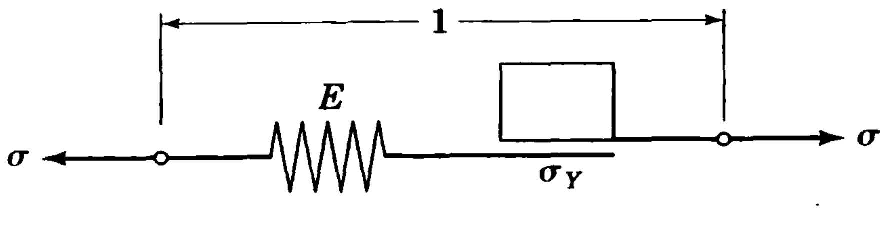

FIGURE 1.1. One-dimensional frictional device illustrating rate–independent plasticity.

a. The total strain ε splits into a part ε^e on the spring with constant E, referred to as the *elastic part*, and a strain ε^p on the friction device referred to as the *plastic part*, that is

$$\boxed{\varepsilon = \varepsilon^e + \varepsilon^p.} \tag{1.2.1}$$

b. By obvious equilibrium considerations, the stress on the spring with constant E is σ, and we have the elastic relationship

$$\boxed{\sigma = E\varepsilon^e \equiv E\left(\varepsilon - \varepsilon^p\right).} \tag{1.2.2}$$

Now we characterize the mechanical response of the friction element as follows.

1.2.1.1 Irreversible frictional response.

Assume that ε, ε^p and σ are functions of time in an interval $\left[0, T\right] \subset \mathbb{R}$. In particular, we let

$$\varepsilon^p : \left[0, T\right] \to \mathbb{R},$$

and

$$\dot{\varepsilon}^p = \frac{\partial}{\partial t}\varepsilon^p. \tag{1.2.3}$$

Change in the configuration of the frictional device is possible only if $\dot{\varepsilon}^p \neq 0$. To characterize this change, we isolate the frictional device as shown in Figure 1.2. We make the following physical assumptions.

1. The stress σ in the frictional device *cannot be greater in absolute value than* $\sigma_Y > 0$. This means that the *admissible stresses* are constrained to lie in the closed interval $[-\sigma_Y, \sigma_Y] \subset \mathbb{R}$. For future use we introduce the notation

$$\boxed{\mathbb{E}_\sigma = \left\{\tau \in \mathbb{R} \mid f(\tau) := |\tau| - \sigma_Y \le 0\right\}} \tag{1.2.4}$$

to designate the set of admissible stresses. For reasons explained below, we denote by σ_Y the *flow stress* of the friction device. The function $f : \mathbb{R} \to \mathbb{R}$, defined as

$$f(\tau) := |\tau| - \sigma_Y \le 0, \tag{1.2.5}$$

then is referred to as the *yield condition*. Note that $\mathbb{E}_\sigma$ is a closed interval and, therefore, it is a *closed convex set*.

2. If the absolute value σ of the applied stress is less than the flow stress σ_Y, *no change* in ε^p takes place, i.e., $\dot{\varepsilon}^p = 0$. This condition implies

$$\dot{\varepsilon}^p = 0 \text{ if } f(\sigma) := |\sigma| - \sigma_Y < 0. \tag{1.2.6}$$

From (1.2.2) and (1.2.6) it follows that

$$f(\sigma) < 0 \Rightarrow \dot{\sigma} = E\dot{\varepsilon}, \tag{1.2.7}$$

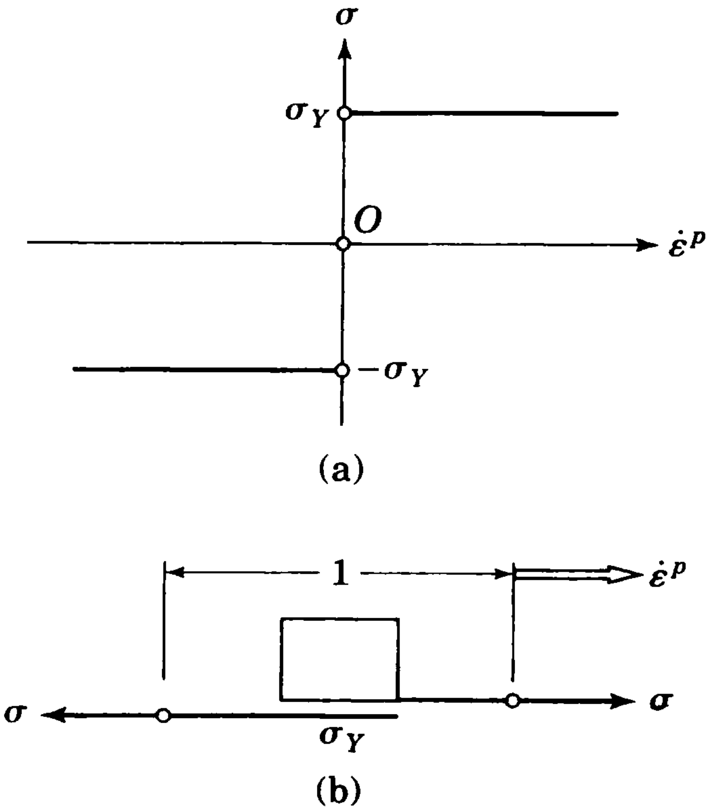

FIGURE 1.2B. Characterization of frictional response for a device with constant $\sigma_Y > 0$.

and the *instantaneous* response of the device is *elastic* with spring constant E. This motivates the denomination of *elastic range* given to the *open* set

$$\operatorname{int}\left(\mathbb{E}_\sigma\right) = \left\{\tau \in \mathbb{R} \mid f(\tau) := |\tau| - \sigma_Y < 0\right\}, \tag{1.2.8}$$

since (1.2.6) and (1.2.7) hold for $\sigma \in \operatorname{int}\left(\mathbb{E}_\sigma\right)$.

3. Because, by assumption 1, stress states σ such that $f(\sigma) = |\sigma| - \sigma_Y > 0$ are *inadmissible* and $\dot{\varepsilon}^p = 0$ for $f(\sigma) < 0$ by assumption 2, a ***change in*** ε^p ***can take place only if*** $f(\sigma) = |\sigma| - \sigma_Y = 0$. If the latter condition is met, the frictional device experiences ***slip in the direction of the applied stress*** σ, with ***constant slip rate***. Let $\gamma \geq 0$ be the absolute value of the slip rate. Then the preceding physical assumption takes the form

$$\left.\begin{aligned} \dot{\varepsilon}^p &= \gamma \geq 0 \quad \text{if} \quad \sigma = \sigma_Y > 0, \\ \dot{\varepsilon}^p &= -\gamma \leq 0 \quad \text{if} \quad \sigma = -\sigma_Y < 0. \end{aligned}\right\} \tag{1.2.9}$$