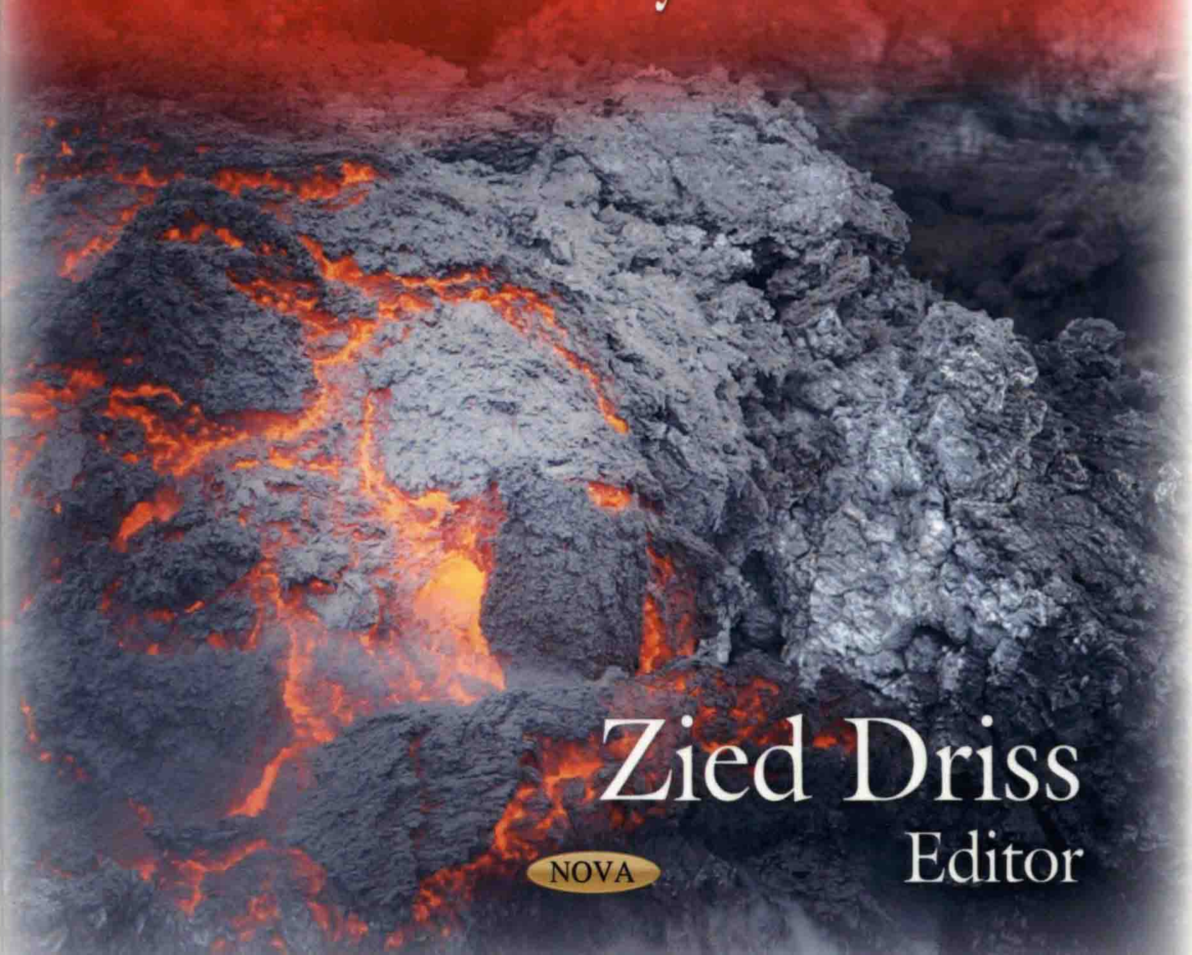
Physics Research and Technology

Turbulent Flows

Prediction, Modeling and Analysis



Zied Driss

Editor

NOVA

PHYSICS RESEARCH AND TECHNOLOGY

TURBULENT FLOWS

PREDICTION, MODELING AND ANALYSIS

ZIED DRISS
EDITOR

nova publishers
New York

For permission to use material from this book please contact us:
Telephone 631-231-7269; Fax 631-231-8175
Web Site: http://www.novapublishers.com

Library of Congress Cataloging-in-Publication Data

ISBN: 978-1-62417-742-2

Published by Nova Science Publishers, Inc. † New York

PHYSICS RESEARCH AND TECHNOLOGY

TURBULENT FLOWS

PREDICTION, MODELING AND ANALYSIS

PHYSICS RESEARCH AND TECHNOLOGY

Additional books in this series can be found on Nova's website under the Series tab.

Additional e-books in this series can be found on Nova's website under the e-books tab.

ENGINEERING TOOLS, TECHNIQUES AND TABLES

Additional books in this series can be found on Nova's website under the Series tab.

Additional e-books in this series can be found on Nova's website under the e-books tab.

PREFACE

"TURBULENT FLOWS: PREDICTION, MODELING AND ANALYSIS"

Turbulent flow means fluid flow in which the fluid undergoes irregular fluctuations. Understanding the turbulent behavior in flowing fluids is one of the most intriguing and important problems that have been the focus of research for decades due to its great importance in a variety of engineering applications. Common examples of turbulent flow include atmospheric and ocean currents, flow through turbines and pumps, blood flow in arteries, oil transport in pipelines, lava flow, and the flow in boat wakes and around aircraft wing tips, etc. Moreover, studying and understanding of turbulence is necessary to comprehend the flow of blood in the heart, especially in the left ventricle, where the movement is particularly swift. A solid grasp of turbulence, can also reduce the aerodynamic drag on an automobile or a commercial airliner, increase the maneuverability of a jet fighter or improve the fuel efficiency of an engine.

In the present book, we focus on areas of current turbulence research. Recent progress on modeling and analysis of turbulent flows is reviewed, and likely directions for future research on these topics are indicated. This text is unusual in as much as it provides both general commentaries as well as recent specialized developments in the field of turbulence modeling. As such it provides access to both historically validated and accepted results and newer ideas and approaches to the problem of modeling turbulence. Also of interest, we chose to devote a fraction of this book to recent developments in stability of parallel shear flows which will be useful to study eventual transition to turbulence.

This book is unique in that it provides a balanced perspective with an emphasis on both rigorous mathematical developments and a focus on engineering problems in a field traditionally dominated by empiricism.

We sincerely hope that the reader will find our choice of material interesting and stimulating. The book is foremost intended for researchers and graduate students with a basic knowledge of fundamental fluid dynamics. We particularly hope it will help young researchers at the beginning of their scientific careers to gain an overview as well as detailed knowledge of the recent developments in the field.

Many colleagues have generously provided comments and material from their past and current research. We especially wish to thank all of the authors who submitted chapters at our requests.

Editors

CONTENTS

In: Turbulent Flows: Prediction, Modeling and Analysis
Editor: Zied Driss
ISBN: 978-1-62417-742-2

Chapter 1

ON ANALYTICAL SOLUTIONS TO THE THREE-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH GENERAL FORCING FUNCTIONS AND THEIR RELATION TO TURBULENCE

Gunawan Nugroho*

Department of Engineering Physics, Institut Teknologi Sepuluh Nopember
Jl. Arief Rahman Hakim, Sukolilo, Surabaya, Indonesia

ABSTRACT

The three-dimensional incompressible viscous flows are solved in this work with the application of the transformed coordinate which defines as a set of functionals, $h_i(\xi) = k_i x + l_i y + m_i z - c_i t$. The solution is proposed from the base of general Riccati equation, which is firstly substituted into the Navier-Stokes equations to produce the polynomial equation with variable coefficients. The resultant solutions from the system of Riccati and polynomial are then evaluated by the proposed method of integral evaluation. The simulation shows the fluctuation of the decaying velocity through time due to energy dissipation. The rapid velocity accumulations are also detected providing that the solutions may produce singularity in the small scale of turbulent flows. The bifurcation is then detected which revealed the strong nonlinearity in the small scale of turbulent flows.

Keywords: continuity equation, the navier-stokes equations, analytical solutions, partial differential equations, integral evaluation

MSC Number (2010): 35A09, 35C05, 76D05

* Email address: gunawan@ep.its.ac.id, gunawanzz@yahoo.com

1. INTRODUCTION

The problem of searching for the classes of analytical solutions of the full Navier-Stokes equations is highly demanding from both theoretical and practical viewpoints, as has been described in the literature [1]. Analytical solutions often play a special role in the theory of nonlinear equations and they are found able to describe the detailed behaviour of the concerning systems [2]. Analytical solutions also facilitate a theoretical understanding, paving the way to global solutions. They may help explain the issue of global smoothness in time [3]. Moreover, the solutions may be examined as models for turbulence [4]. Also, some particular solutions such as vortex solutions play a significant role in the development of turbulence theories [5].

The main difficulty of analytical solution of the Navier-Stokes equations is the contribution of the nonlinear terms representing fluid inertia which then troubled the conventional analysis in general cases. However, it is not surprising that they attract many efforts until recently, since the full solution of the three-dimensional Navier-Stokes equations still remains as one of the open problems in mathematical physics, especially in the sense of functional analysis [6]. The most important developments can be found in the review papers or in the monograph as in [7,8]. As one of the most interesting is a numerically based existence theorem which then can be established the properties of the solution [9]. By taking that numerical solutions converge to analytical solutions, the similar procedure can then be applied.

Concerning to the analytical solutions, there are some works have already been conducted in the literatures [10,11,12]. As in the most cases, analytical solutions are examined only in special conditions in which the nonlinearity are weakened or even removed from the analysis. The type of the simplified analysis is applicable for steady and unsteady Couette and Poiseuille flows in which the nonlinear terms are removed permanently [13]. The other less known example is applicable to Beltrami flows in which the nonlinear terms are nonzero in the Navier-Stokes equations but they fade in the vorticity equations [14].

However, more sophisticated analysis of the Navier-Stokes equations is also conducted and gives more insight to the problems. One of them is the transformation of the Navier-Stokes equations to the Schrodinger equation, performed by application of the Riccati equation [15].

It has good prospects since the Schrodinger equation is linear and has well defined solutions. The method of Lie group theory is also applied in order to transform the original partial differential equations into ordinary differential systems [16].

It is concluded that an approximate series solution is obtained. The same route is taken by Meleshko [17] and by Thailert [18], in transforming the Navier-Stokes equations to solvable linear systems. Furthermore, less popular methods, such as the Hodograph-Legendre transformation, have also been applied to reduce the original problem to one more tractable, and thus closer to the goal of obtaining analytical solutions [19]. The reduction of the full set of Navier-Stokes equations to be a class of nonlinear ordinary differential equation is also performed [20]. The solution applied to both zero and constant pressure gradient cases. The method of introducing special solutions for velocity has also been investigated in [21,22].

The connection of the solutions with turbulence is discussed intensively by investigating that global in time continuation is still unsolvable for three-dimensional flows [25,26].

The question arises as if such singularities exist, they might be related to turbulence by invoking that we have global smooth solution for two-dimensional flows, and turbulence is three-dimensional phenomena. This hypothesis, however, has serious difficulties as the observed phenomena are so far bounded in nature.

In that case, different viewpoint is proposed. The argument that turbulent solutions should have no singularities is supported based on the triviality of the solution for simple energy equation. Analysis of global trivial solutions is important from mathematical and physical aspects, it has wide application and may be correlated to many areas where some hierarchical solutions are arranged. Classical procedure of vector identities is implemented for producing trivial solutions. Violation from trivial solutions is also investigated. Investigation of nontrivial solutions is related to the rate of energy generated or destroyed. The nontrivial solutions from the Navier-Stokes equations are then established as turbulent solutions due to energy accumulations.

Apart from the simplified analysis, the current research method is also inspired by the importance of the Burger equation which is able to describe one-dimensional turbulence.

The Burger equation can be viewed as a Riccati equation in one-dimensional case which the solution is mathematically well-posed and physically relevant for showing the collection of shocks, decaying velocity and its profile in nozzle flows even though it is not very accurate for three-dimensional cases.

However, the equation provided us with the mathematical insight and physical understanding on the nature of intermittency of turbulence. In this work, the mathematical formulations by implementing the Riccati equation are developed more. The procedure for obtaining the solutions are derived in a quite simple way which is based on the implementation of coordinate transformation of a set of traveling wave ansatzs. The resulting expression is then produced polynomial equation which then combined with the solution of the Riccati to form final solutions.

2. PROBLEM FORMULATION

The problem of searching the analytical solutions to the Navier-Stokes equations is a challenging task. The equations are usually simplified and solved concerning to the specific problems. However, our contribution is to evaluate the scheme for general incompressible flow cases in cartesian coordinate.

Consider the continuity and the three-dimensional incompressible Navier-Stokes equations with external forces as in the following form,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (1a)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial x^2}+\upsilon\frac{\partial^2 u}{\partial y^2}+\upsilon\frac{\partial^2 u}{\partial z^2}+F_1 \quad (1b)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\frac{\partial^2 v}{\partial x^2}+\upsilon\frac{\partial^2 v}{\partial y^2}+\upsilon\frac{\partial^2 v}{\partial z^2}+F_2 \quad (1c)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial^2 w}{\partial z^2}+F_3 \tag{1d}$$

where p is static pressure, ρ is fluid density, υ is kinematic viscosity and $\nabla=(\partial/\partial x,\partial/\partial y,\partial/\partial z)$. All solutions describe the three velocity components in the three spatial directions, i.e., $u=u(x,y,z,t)$, $v=v(x,y,z,t)$ and $w=w(x,y,z,t)$.

The presence of forcing functions F_i is physically relevant and can be widely observed in atmospheric flows, which the flow under the presence of the coriolis and electromagnetic forces.

Beginning the process, the following relation is produced from the continuity equation (1a),

$$u=-\int_x\left(v_y+w_z\right)dx+K_1\left(y,z,t\right) \tag{2}$$

2.1. The Formulation for v Velocity and the Pressure Relation

Equation (2) can be substituted into (1d) to give,

$$w_t+\left[-\int_x\left(v_y+w_z\right)dx+K_1\right]w_x+vw_y+ww_z=-\frac{1}{\rho}p_z+\upsilon\left(w_{xx}+w_{yy}+w_{zz}\right)+F_3 \tag{3a}$$

Then the pressure relation can be determined as,

$$p=\rho\int_z\left\{F_3+\upsilon\left(w_{xx}+w_{yy}+w_{zz}\right)+w_x\left[\int_x\left(v_y+w_z\right)dx-K_1\right]-w_t-vw_y-ww_z\right\}dz+K_2\left(x,y,t\right) \tag{3b}$$

The next step is implementing equation (2) and (3b) into (1c),

$$\begin{aligned}&v_x\int_x\left(v_y+w_z\right)dx-K_1v_x=v_t+vv_y+wv_z-\upsilon\left(v_{xx}+v_{yy}+v_{zz}\right)+K_{2y}\\&+\frac{\partial}{\partial y}\left[\int_z\left\{F_3+\upsilon\left(w_{xx}+w_{yy}+w_{zz}\right)+w_x\left[\int_x\left(v_y+w_z\right)dx-K_1\right]-w_t-vw_y-ww_z\right\}dz\right]\end{aligned} \tag{4a}$$

Suppose that the coordinate also satisfy the following set of traveling wave ansatz,

$$h_i\left(\xi\right)=k_ix+l_iy+m_iz-c_it \tag{4b}$$

where k_i,l_i,m_i and c_i are constants.

The step is now transforming the cartesian coordinate into ξ-coordinate by setting the subscript $i=1,2,3,4$ such that we have four equations for the coordinate transformation. The results of $x=\psi_1(\xi),y=\psi_2(\xi),z=\psi_3(\xi)$ and $t=\psi_4(\xi)$ are then determined. Applying one of the traveling wave ansatzs, i.e. $h_1(\xi)=k_1x+l_1y+m_1z-c_1t$ into (4a), the considered equation then transformed into,

$$\frac{k_1}{h_{1\xi}}v_\xi\left(klv+kmw\right)+\frac{k_1}{h_{1\xi}}K_1v_\xi=-\frac{c_1}{h_{1\xi}}v_\xi+\frac{l_1}{h_{1\xi}}vv_\xi+\frac{m_1}{h_{1\xi}}wv_\xi-\upsilon\left(\frac{k_1^2}{h_{1\xi}^2}+\frac{l_1^2}{h_{1\xi}^2}+\frac{m_1^2}{h_{1\xi}^2}\right)v_{\xi\xi}+\frac{l_1}{m_1}F_3+F_2+\frac{l_1}{h_{1\xi}}K_{2\xi}$$
$$+\upsilon\left(\frac{k_1^2l_1}{h_{1\xi}^2m_1}+\frac{l_1^3}{h_{1\xi}^2m_1}+\frac{l_1m_1^2}{h_{1\xi}^2m_1}\right)w_{\xi\xi}+\frac{k_1}{h_{1\xi}}w_\xi\left[\left(k_1\frac{l_1^2}{m_1}v+k_1l_1w\right)-K_1\right]+\frac{c_1l_1}{h_{1\xi}m_1}w_\xi-\frac{l_1^2}{h_{1\xi}m_1}vw_\xi-\frac{l_1}{h_{1\xi}}ww_\xi \tag{4c}$$

Rearranging (4c) to give,

$$a_1v_{\xi\xi}+a_2vv_\xi+\left(a_3+a_4w\right)v_\xi+a_5w_\xi v=a_6w_{\xi\xi}+a_7w_\xi+a_8ww_\xi+\frac{l_1}{m_1}F_3+F_2+\frac{l_1}{h_{1\xi}}K_{2\xi} \tag{4d}$$

where a_i are variables that depend on ξ.

Lemma 1: Equation (4d) is transformable into the system of Riccati and third order polynomial equations.

Proof: Consider the Riccati equation,

$$v_\xi=b_1v^2+b_2v+b_3 \tag{5a}$$

such that the following relation is satisfied,

$$v_{\xi\xi}=b_{1\xi}v^2+2b_1vv_\xi+b_{2\xi}v+b_2v_\xi+b_{3\xi}$$
$$v_{\xi\xi}=2b_1^2v^3+\left(b_{1\xi}+3b_1b_2\right)v^2+\left(b_{2\xi}+2b_1b_3+b_2^2\right)v+b_{3\xi}+b_2b_3 \tag{5b}$$

Substituting into (4d) to get the following polynomial equation,

$$\begin{gathered}\left(2a_1b_1^2+a_2b_1\right)v^3+\left(a_1b_{2\xi}+2a_1b_1b_3+a_1b_2^2+a_2b_3+a_3b_2+a_5w_\xi+a_4b_2w\right)v+\\ \left(a_1b_{1\xi}+3a_1b_1b_2+a_2b_2+a_3b_1+a_4b_1w\right)v^2=\\ a_6w_{\xi\xi}+a_7w_\xi+a_8ww_\xi-a_4b_3w-a_1b_{3\xi}-a_1b_2b_3-a_3b_3+\frac{l_1}{m_1}F_3+F_2+\frac{l_1}{h_{1\xi}}K_{2\xi}\end{gathered} \tag{5c}$$

Therefore, equation (4d) is transformed to the system of (5a) and (5c).

This proves lemma 1.

Set $\left(2a_1b_1^2+a_2b_1\right)=0$ and $\left(a_1b_{1\xi}+3a_1b_1b_2+a_2b_2+a_3b_1+a_4b_1w\right)=0$, thus the expression for b_1 and b_2 are defined by,

$$b_1 = -\frac{a_2}{2a_1} \tag{5d}$$

$$b_2 = -\frac{\left(a_1 b_{1\xi} + a_3 b_1 + a_4 b_1 w\right)}{3a_1 b_1 + a_2} \tag{5e}$$

Therefore, equation (5c) is reduced into,

$$v = \frac{a_6 w_{\xi\xi} + a_7 w_\xi + a_8 w w_\xi - a_4 b_3 w - a_1 b_{3\xi} - a_1 b_2 b_3 - a_3 b_3 + \frac{l_1}{m_1} F_3 + F_2 + \frac{l_1}{h_{1\xi}} K_{2\xi}}{a_1 b_{2\xi} + 2a_1 b_1 b_3 + a_1 b_2^2 + a_2 b_3 + a_3 b_2 + a_5 w_\xi + a_4 b_2 w} \tag{5f}$$

The expression for b_3 will be determined later as a requirement of unique solutions under general initial-boundary values. Also note that the solution for the system (5a) and (5f) will be similar to that of the velocity in z direction as derived in the subsequent paragraph.

2.2. The Formulation for w Velocity

The step now is performing equations (2) and (3b) into (1b),

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\int_x \left(v_y + w_z\right)dx - K_1\right] + \left[-\int_x \left(v_y + w_z\right)dx + K_1\right]\left(v_y + w_z\right) + v\frac{\partial}{\partial y}\left[\int_x \left(v_y + w_z\right)dx - K_1\right] \\
&+ w\frac{\partial}{\partial z}\left[\int_x \left(v_y + w_z\right)dx - K_1\right] = \frac{\partial}{\partial x}\left[\int_z \left\{\begin{matrix} F_3 + \upsilon\left(w_{xx} + w_{yy} + w_{zz}\right) + \\ w_x\left[\int_x \left(v_y + w_z\right)dx - K_1\right] - w_t - v w_y - w w_z \end{matrix}\right\} dz\right] \\
&+ K_{2x} + \upsilon\left(\left(v_{xy} + w_{xz}\right) + \frac{\partial^2}{\partial y^2}\left[\int_x \left(v_y + w_z\right)dx - K_1\right] + \frac{\partial^2}{\partial z^2}\left[\int_x \left(v_y + w_z\right)dx - K_1\right]\right) + F_1
\end{aligned} \tag{6a}$$

Performing the coordinate transformation (4b),

$$\begin{aligned}
&-\frac{c_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) + \left[-\left(k_1 l_1 v + k_1 m_1 w\right) + K_1\right]\left(\frac{l_1}{h_{1\xi}} v_\xi + \frac{m_1}{h_{1\xi}} w_\xi\right) + v\frac{l_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) + \\
&w\frac{m_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) = \frac{k_1}{h_{1\xi}} K_{2\xi} + \upsilon\left(\frac{k_1 l_1}{h_{1\xi}^2} v_{\xi\xi} + \frac{k_1 m_1}{h_{1\xi}^2} w_{\xi\xi}\right) + \upsilon\frac{l_1^2}{h_{1\xi}^2}\left(k_1 l_1 v_{\xi\xi} + k_1 m_1 w_{\xi\xi} - K_1\right) + \\
&\frac{k_1}{m_1}\left\{F_3 + \upsilon\left(\frac{k_1^3}{h_{1\xi}^2 m_1} + \frac{k_1 l_1^2}{h_{1\xi}^2 m_1} + \frac{k_1 m_1}{h_{1\xi}^2}\right) w_{\xi\xi} + \frac{k_1}{h_{1\xi}} w_\xi\left[\left(\frac{k_1^2 l_1}{m_1} v + k_1^2 w\right) - K_1\right] + \frac{k_1 c_1}{h_{1\xi} m_1} w_\xi - \frac{k_1 l_1}{h_{1\xi} m_1} v w_\xi - \frac{k_1}{h_{1\xi}} w w_\xi\right\} + \\
&\upsilon\frac{m_1^2}{h_{1\xi}^2}\left(k_1 l_1 v_{\xi\xi} + k_1 m_1 w_{\xi\xi} - K_1\right) + F_1
\end{aligned} \tag{6b}$$

Thus by grouping the above equation, the momentum in z direction is given by,

$$a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - \left(\frac{c_1}{h_{1\xi}} + \upsilon \frac{l_1^2}{h_{1\xi}^2} + \upsilon \frac{m_1^2}{h_{1\xi}^2} \right) K_1 =$$
$$a_{12} v_{\xi\xi} + a_{13} v v_\xi + \left(a_{14} + a_{15} w\right) v_\xi + a_{16} w_\xi v \tag{6c}$$

Since equation (6c) is similar to that of (4d), it is reasonable to state that the solution for v is also similar to (5f).

Repeating the procedure described by (5a – f) with different variables, b_4, b_5 and b_6 yielding,

$$v = \frac{a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi - a_{15} b_6 w - a_{12} b_{6\xi} - a_{12} b_5 b_6 - a_{14} b_6 + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - \left(\frac{c_1}{h_{1\xi}} + \upsilon \frac{l_1^2}{h_{1\xi}^2} + \upsilon \frac{m_1^2}{h_{1\xi}^2} \right) K_1}{a_{12} b_{5\xi} + 2 a_{12} b_4 b_6 + a_{12} b_5^2 + a_{13} b_6 + a_{14} b_5 + a_{16} w_\xi + a_{15} b_5 w} \tag{6d}$$

Equating (5f) and (6d) to get,

$$v = \frac{a_6 w_{\xi\xi} + a_7 w_\xi + a_8 w w_\xi - a_4 b_3 w - a_1 b_{3\xi} - a_1 b_2 b_3 - a_3 b_3 + \frac{l_1}{m_1} F_3 + F_2 + \frac{l_1}{h_{1\xi}} K_{2\xi}}{a_1 b_{2\xi} + 2 a_1 b_1 b_3 + a_1 b_2^2 + a_2 b_3 + a_3 b_2 + a_5 w_\xi + a_4 b_2 w}$$
$$= \frac{a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi - a_{15} b_6 w - a_{12} b_{6\xi} - a_{12} b_5 b_6 - a_{14} b_6 + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - \left(\frac{c_1}{h_{1\xi}} + \upsilon \frac{l_1^2}{h_{1\xi}^2} + \upsilon \frac{m_1^2}{h_{1\xi}^2} \right) K_1}{a_{12} b_{5\xi} + 2 a_{12} b_4 b_6 + a_{12} b_5^2 + a_{13} b_6 + a_{14} b_5 + a_{16} w_\xi + a_{15} b_5 w} \tag{6e}$$

which then can be performed algebraically to form a single expression.

3. The Analytical Solutions

In this research, we are interested to the class of solutions that is physically important [23,24],

$$w = r(\xi) e^{i h_1(\xi)} \tag{7a}$$

Substituting (7a) into (6e) to get,

$$r_\xi \left(a_{17} r_{\xi\xi} + a_{18} r r_\xi + a_{19} r_\xi + a_{20} r + a_{21} \right) + r \left(a_{22} r_{\xi\xi} + a_{23} r r_\xi + a_{24} r_\xi + a_{25} r + a_{26} \right)$$
$$+ a_{27} r_{\xi\xi} + a_{28} r r_\xi + a_{29} r_\xi + a_{30} r + a_{31} = 0 \tag{7b}$$

By setting,

$$r_{\xi} = b_7 r^2 + b_8 r + b_9 \tag{7c}$$

such that the following relation is fulfilled,

$$r_{\xi\xi} = 2b_7^2 r^3 + \left(b_{7\xi} + 3b_7 b_8\right) r^2 + \left(b_{8\xi} + 2b_7 b_9 + b_8^2\right) r + b_{9\xi} + b_8 b_9 \tag{7d}$$

Thus, applying (7c) and (7d) into (7b), the resulting polynomial equation is defined by,

$$a_{32} r^5 + a_{33} r^4 + a_{34} r^3 + a_{35} r^2 + a_{36} r + a_{37} = 0 \tag{7e}$$

where a_{37} is given by,

$$a_{37} = a_{17}\left(b_{9\xi} b_9 + b_8 b_9^2\right) + a_{19} b_9^2 + a_{21} b_9 + a_{27}\left(b_{9\xi} + b_8 b_9\right) + a_{29} b_9 + a_{31} \tag{7f}$$

Set $a_{37} = 0$ then b_8 can be obtain as a function of b_9.

This procedure will reduced equation (7e) to the fourth order polynomial which the root is solvable by radicals, say be written as $r = \beta(\xi)$. Meanwhile, b_7 will be defined later by the similar condition applied to b_3.

3.1. The Solution for the Riccati Equation

In order to solve the system of (7c) and (7e), the following step is necessary [25]

Lemma 2: Consider equation (7c) and set $b_8 = f_1 - \frac{f_{2\xi}}{f_2}$ to generate,

$$Z_{\xi} = \frac{b_7}{C_1 f_2} Z^2 + f_1 Z + C_1 f_2 b_9$$

where $Z = C_1 f_2 r$. There exists a function α such that $f_2 b_9 = \alpha Z$, which generate the Bernoulli equation. The Riccati equation then has a closed-form exact solution when α is solvable.

Proof: Set $b_8 = f_1 - \frac{f_{2\xi}}{f_2}$ to rearrange equation (7c) as,

$$r_{\xi} + \frac{f_{2x}}{f_2} r = \frac{1}{C_1 f_2}\left(C_1 f_2 r\right)_{\xi} = b_7 r^2 + f_1 r + b_9 \tag{8a}$$

where C_1 is a constant. Suppose that $Z = C_1 f_2 r$, then the following equation is produced,