Numerical Approximation of Partial Differential Equations

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偏微分方程的数值近似法

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A Fulvia e Tiziana

Preface

Everything is more simple than one thinks
but at the same time more complex than one can understand
Johann Wolfgang von Goethe

To reach the point that is unknown to you,
you must take the road that is unknown to you
St. John of the Cross

This is a book on the numerical approximation of partial differential equations (PDEs). Its scope is to provide a thorough illustration of numerical methods (especially those stemming from the variational formulation of PDEs), carry out their stability and convergence analysis, derive error bounds, and discuss the algorithmic aspects relative to their implementation.

A sound balancing of theoretical analysis, description of algorithms and discussion of applications is our primary concern.

Many kinds of problems are addressed: linear and nonlinear, steady and time-dependent, having either smooth or non-smooth solutions. Besides model equations, we consider a number of (initial-) boundary value problems of interest in several fields of applications.

Part I is devoted to the description and analysis of general numerical methods for the discretization of partial differential equations.

A comprehensive theory of Galerkin methods and its variants (Petrov-Galerkin and generalized Galerkin), as well as of collocation methods, is developed for the spatial discretization. This theory is then specified to two numerical subspace realizations of remarkable interest: the finite element method (conforming, non-conforming, mixed, hybrid) and the spectral method (Legendre and Chebyshev expansion).

For unsteady problems we will illustrate finite difference and fractional-step schemes for marching in time. Finite differences will also be extensively considered in Parts II and III in the framework of convection-diffusion problems and hyperbolic equations. For the latter we will also address, briefly, the schemes based on finite volumes.

For the solution of algebraic systems, which are typically very large and sparse, we revise classical and modern techniques, either direct and iterative with preconditioning, for both symmetric and non-symmetric matrices. A short account will be given also to multi-grid and domain decomposition methods.

Parts II and III are respectively devoted to steady and unsteady problems. For each (initial-) boundary value problem we consider, we illustrate the main theoretical results about well-posedness, i.e., concerning existence, uniqueness and a-priori estimates. Afterwards, we reconsider and analyze the previously mentioned numerical methods for the problem at hand, we derive the corresponding algebraic formulation, and we comment on the solution algorithms.

To begin with, we consider all classical equations of mathematical physics: elliptic equations for potential problems, parabolic equations for heat diffusion, hyperbolic equations for wave propagation phenomena. Furthermore, we discuss extensively advection-diffusion equations for passive scalars and the Navier-Stokes equations (together with their linearized version, the Stokes problem) for viscous incompressible flows. We also derive the equations of fluid dynamics in their general form.

Unfortunately, the limitation of space and our own experience have resulted in the omission of many important topics that we would have liked to include (for example, the Saint-Venant model for shallow water equations, the system of linear elasticity and the biharmonic equation for membrane displacement and thin plate bending, the drift-diffusion and hydrodynamic models for semiconductor devices, the Navier-Stokes and Euler equations for compressible flows).

This book is addressed to graduate students as well as to researchers and specialists in the field of numerical simulation of partial differential equations.

As a graduate text for Ph.D. courses it may be used in its entirety. Part I may be regarded as a one quarter introductory course on variational numerical methods for PDEs. Part II and III deal with its application to the numerical approximation of time-independent and time-dependent problems, respectively, and could be taught through the two remaining quarters. However, other solutions may work well. For instance, supplementing Part I with Chapters 6, 11 and most part of 14 may be suitable for a one semester course. The rest of the book could be covered in the second semester. Following a different key, Part I plus Chapters 8, 9, 10, 12, 13 and 14 can be regarded as an introduction to numerical fluid dynamics. Other combinations are also envisageable.

The authors are grateful to Drs. C. Byrne and J. Heinze of Springer-Verlag for their encouragement throughout this project. The assistence of the technical staff of Springer-Verlag has contributed to the final shaping of the manuscript.

This book benefits from our experience in teaching these subjects over the past years in different academical institutions (the University of Minnesota at Minneapolis, the Catholic University of Brescia and the Polythecnic of Milan for the first author, the University of Trento for the second author), and from students' reactions. Help was given to us by several friends and collaborators who read parts of the manuscript or provided figures or tables. In this connection we are happy to thank V.I. Agoshkov, Yu.A. Kuznetsov, D. Ambrosi, L. Bergamaschi, S. Delladio, M. Manzini, M. Paolini, F. Pasquarelli, L. Stolcis, E. Zampieri, A. Zaretti and in particular C. Bernini, P. Gervasio and F. Saleri.

We would also wish to thank Ms. R. Holliday for having edited the language of the entire manuscript. Finally, the expert and incredibly adept typing of the TEX-files by Ms. C. Foglia has been invaluable.

Milan and Trento
May, 1994

Alfio Quarteroni
Alberto Valli

In the second printing of this book we have corrected several misprints, and introduced some modifications to the original text.

More precisely, we have sligthly changed Sections 2.3.4, 3.4.1, 8.4 and 12.3, and we have added some further comments to Remark 8.2.1.

We have also completed the references of those papers appeared after 1994.

Milan and Trento
December, 1996

Alfio Quarteroni
Alberto Valli

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1. Introduction

Numerical approximation of partial differential equations is an important branch of Numerical Analysis. Often, it demands a knowledge of many aspects of the problem.

First of all, the physical background of the problem is required in order to understand the behaviour of expected solutions. This may often lead to the choice of convenient numerical methods.

Secondly, modern formulation of the problem based on the variational (weak) form ought to be considered, as it allows the search for generalized solutions in Hilbert (or Banach) functional spaces. Variational techniques yield a-priori estimates for the solution, which in turn indicate in which kind of norms any virtual numerical solution can be proven to be stable. Furthermore, results about smoothness of the mathematical solutions may suggest the numerical methodology to be used, and consequently, determine the kind of accuracy that can be achieved. The latter is pointed out from the error analysis.

Clearly, specific attention should be paid to the algorithmic aspects concerned with the choice of any numerical method.

This book aims at providing general ideas on numerical approximation of partial differential equations, although (obviously) not all possible existing methods will be considered. In this respect, we mainly focus on variational numerical methods for the discretization of space derivatives, and on finite difference and fractional-step methods for advancing, in time, unsteady problems.

Whenever possible, we present the unifying approach behind a-priori different numerical strategies, provide general theory for analysis and illustrate a variety of algorithms that can be used to compute the effective numerical solution of the problem at hand, taking into consideration its algebraic structure. Consequently, we try to avoid using technicalities (or tricks, or algorithms) that work only in very specific situations, or that are not sustained from a sound theoretical background. Some problems (and methods) are discussed on a case-to-case basis, but very often they are included in a single logical unit (say Chapter, or Section).

1.1 The Conceptual Path Behind the Approximation

We consider a great number of mathematical problems, and numerical methods for their solution. For the approximation of any given boundary value problem, we schematically illustrate in Fig. 1.1.1 the decision path that needs to be followed.

Level [1] is the boundary value problem at hand under its weak formulation accounting for the prescribed boundary conditions.

Level [2] provides the kind of discretization (or numerical method) that can be pursued in order to reduce the given problem to one having finite dimension. Of course, the strategy adopted will determine the structure of the numerical problem.

Throughout this book we mainly consider two kinds of discretization. The former is the Galerkin method, together with its remarkable variant, the Petrov-Galerkin method, which is based on an integral formulation of the differential problem. The second discretization we consider, is the collocation method, which is, instead, based on the fulfillment of the differential equations at some selected points of the computational domain. We then reformulate the collocation method under a generalized Galerkin mode, precisely combining the Galerkin approach with numerical evaluation of integrals using Gaussian formulae.

At a lower extent, we will address finite difference schemes for space discretization, especially for nonlinear convection-diffusion equations and for problems of wave propagation. For the latter we will also present the approach based on the finite volume method, which is very popular in computational fluid dynamics.

Finally, we will illustrate shortly the elementary principles of the domain decomposition method, an approach which offers the best promise for the parallel solution of large problems in the field of scientific computing.

Other approaches are often encountered in the literature as well, but they will only be addressed incidentally in this book.

Level [3] specifies the nature of the subspaces used in the approximation. Typically, we have piecewise-polynomial functions of low degree when using finite elements, and global algebraic polynomials of high degree for spectral methods. These two remarkable cases will be discussed and analyzed in some of their variants (mixed finite elements, Legendre and Chebyshev spectral collocation methods). The choice operated at this level determines the functional structure of the numerical solution, the kind of accuracy that can be achieved, besides affecting the topological form of the resulting algebraic system.

At level [4] the selection of convenient algorithms needs to be accomplished to solve the algebraic problem, exploiting, at most, the topological structure and the properties of the associated matrices. We illustrate all the important methods available nowadays for solving large scale symmetric and