Advanced Courses in Mathematics CRM Barcelona

Masayuki Asaoka
Aziz El Kacimi Alaoui
Steven Hurder
Ken Richardson

Foliations: Dynamics, Geometry and Topology

CENTRE DE RECERCA MATEMÀTICA

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Editors for this volume:
Jesús Álvarez López (Universidade de Santiago de Compostela)
Marcel Nicolau (Universitat Autònoma de Barcelona)

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Masayuki Asaoka
Department of Mathematics
Kyoto University
Kyoto, Japan

Aziz El Kacimi Alaoui
LAMAV-ISTV2
Université de Valenciennes
Valenciennes, France

Steven Hurder
Department of Mathematics
University of Illinois at Chicago
Chicago, IL, USA

Ken Richardson
Department of Mathematics
Texas Christian University
Fort Worth, TX, USA

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Carles Casacuberta

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Foreword

This book is an introduction to several active research topics in Foliation Theory. It is based on lecture notes of some of the courses given at the school *Advanced Course on Foliations: Dynamics, Geometry, Topology*, held in May 2010 at the Centre de Recerca Matemàtica (CRM) in Bellaterra, Barcelona. This school was one of the main activities of the Research Programme on Foliations, which took place at the CRM from April to July 2010. The program of that event consisted of five courses taught by Aziz El Kacimi-Alaoui, Steven Hurder, Masayuki Asaoka, Ken Richardson, and Elmar Vogt.

These courses dealt with different aspects of Foliation Theory, which is the qualitative study of differential equations on manifolds. It was initiated by the works of H. Poincaré and I. Bendixson, and later developed by C. Ehresmann, G. Reeb, A. Haefliger, S. Novikov, W. Thurston and many others. Since then, the subject has become a broad research field in Mathematics.

The course of Aziz El Kacimi-Alaoui is an elementary introduction to this theory. Through simple and diverse examples, he discusses, in particular, the fundamental concept of transverse structure.

The lectures of Steven Hurder develop ideas from smooth dynamical systems for the study and classification of foliations of compact manifolds, by alternating the presentation of motivating examples and related concepts. The first two lectures develop the fundamental concepts of limit sets and cycles for leaves, foliation "time" and the leafwise geodesic flow, and transverse exponents and stable manifolds. The third lecture discusses applications of the generalization of Pesin Theory for flows to foliations. The last two lectures consider the classification theory of smooth foliations according to their types: hyperbolic, parabolic or elliptic.

For a smooth locally free action, the collection of the orbits forms a foliation. The leafwise cohomology of the orbit foliation controls the deformation of the action in many cases. The course by Masayuki Asaoka starts with the definition and some basic examples of locally free actions, including flows with no stationary points. After that, he discusses how to compute the leafwise cohomology and how to apply it to the description of deformation of actions.

In the lectures given by Ken Richardson, he investigates generalizations of the ordinary Dirac operator to manifolds endowed with Riemannian foliations or compact Lie group actions. If the manifold comes equipped with a Clifford algebra action on a bundle over the manifold, one may define a corresponding transversal Dirac operator. He studies the geometric and analytic properties of these operators, and obtains a corresponding index formula.

We would like to express our deep gratitude to the authors of these Advanced Courses for their enthusiastic work, to the director, J. Bruna, and the staff of the CRM, whose help was essential in the organization of these Advanced Courses, and to C. Casacuberta, editor of this series, for his help and patience. We also thank the "Ministerio de Educación y Ciencia" and the Ingenio Mathematica programme of the Spanish government for providing financial support for the organization of the courses.

Jesús A. Álvarez López and Marcel Nicolau

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Chapter 1

Deformation of Locally Free Actions and Leafwise Cohomology

Masayuki Asaoka

Introduction

These are the notes of the author's lectures at the *Advanced Course on Foliations* in the research program *Foliations*, which was held at the Centre de Recerca Matemàtica in May 2010. In these notes, we discuss the relationship between deformations of actions of Lie groups and the leafwise cohomology of the orbit foliation.

In the early 1960's, Palais [44] proved the local rigidity of smooth actions of compact groups. Hence, such actions have no non-trivial deformations. In contrast to compact group actions, all known $\mathbb{R}$-actions (i.e., flows) fail to be locally rigid, except for the trivially rigid ones. Moreover, many $\mathbb{R}$-actions change the topological structure of their orbits under perturbation. Their bifurcation is an important issue in the theory of dynamical systems.

In the last two decades, it has been found that there exist locally rigid actions of higher-dimensional Lie groups, and the rigidity theory of locally free actions has undergone a rapid development. The reader can find examples of locally rigid or parameter rigid actions in many papers [5, 9–12, 18, 24, 32, 33, 36, 41, 42, 49, 51–53], some of which will be discussed in this chapter.

A rigidity problem can be regarded as a special case of a deformation problem. In many situations, the deformation space of a geometric structure is described by a system of non-linear partial differential equations. Its linearization defines a cochain complex, called *deformation complex*, and the space of infinitesimal deformations is identified with the first cohomology of this complex. For locally free actions of Lie groups, the deformation complex is realized as the (twisted) leafwise de Rham complex of the orbit foliation.

The reader may wish to develop a general deformation theory of locally free actions in terms of the deformation complex, like the deformation theory of complex manifolds founded by Kodaira and Spencer. However, the leafwise de Rham complex is not elliptic, and this causes two difficulties to develop a fine theory. First, the leafwise cohomology groups are infinite-dimensional in general, and they are hard to compute. Second, we need to apply the implicit function theorem for maps between Fréchet spaces rather than Banach spaces because of *loss of derivatives*. This requires tameness of splitting of the deformation complex, which is hard to prove. Thus, we will focus on techniques to overcome these difficulties in several explicit examples instead of developing a general theory.

The main tools for computation of the leafwise cohomology are Fourier analysis, representation theory, and a Mayer–Vietoris argument developed by El Kacimi Aloui and Tihami. Matsumoto and Mitsumatsu also developed a technique, based on ergodic theory of hyperbolic dynamics. We will discuss these techniques in Section 1.3.

For several actions, the deformation problem can be reduced to a linear one without help of any implicit function theorem, and hence we can avoid a tame estimate of the splitting. In Section 1.4 we will see how to reduce the rigidity problem of such actions to (almost) vanishing of the first cohomology of the leafwise cohomology. The first case is parameter deformation of abelian actions. We will see that the problem is linear in this case. In fact, the deformation space can be naturally identified with the space of infinitesimal deformations. The second case is parameter rigidity of solvable actions. Although the problem itself is not linear in this case, we can decompose it into the solvability of linear equations for several examples.

For general cases, the deformation problem cannot be reduced to a linear one directly. One way to describe the deformation space is to apply Hamilton's implicit function theorem. As mentioned above, this requires a tame estimate on solutions of partial differential equations and is difficult to establish it in general. However, there are a few examples to which we can apply the theorem. Another way is to use the theory of hyperbolic dynamics. We offer a brief discussion of these techniques in Section 1.5.

The author recommends to the readers the survey papers [7] and [39]. The former contains a nice exposition of applications of Hamilton's implicit function theorem to rigidity problems of foliations. The second is a survey about the parameter rigidity problem, which is one of the sources of the author's lectures at the Centre de Recerca Matemàtica.

To end the Introduction, I would like to thank the organizers of the CRM research program *Foliations* for inviting me to give these lectures in the program, and the staff of the CRM for their warm hospitality. I am also grateful to Marcel Nicolau and Nathan dos Santos for many suggestions to improve the notes.

1.1 Locally free actions and their deformations

In this section we define locally free actions and their infinitesimal correspondent. We also introduce the notion of deformation of actions and several concepts of finiteness of codimension of the conjugacy classes of an action in the space of locally free actions.

1.1.1 Locally free actions

In these notes, we will work in the C^∞-category. So, the term *smooth* means C^∞, and all diffeomorphisms are of class C^∞. All manifolds and Lie groups in these notes will be connected. For manifolds M_1 and M_2, we denote the space of smooth maps from M_1 to M_2 by $C^\infty(M_1, M_2)$. It is endowed with the C^∞ compact-open topology. By $\mathcal{F}(x)$ we denote the leaf of a foliation $\mathcal{F}$ which contains a point x.

Let G be a Lie group and M a manifold. We denote the unit element of G by 1_G and the identity map of M by Id_M. We say that a smooth map $\rho\colon M \times G \to M$ is *a (smooth right) action* if

(1) $\rho(x, 1_G) = x$ for all $x \in M$, and

(2) $\rho(x, gh) = \rho(\rho(x, g), h)$ for all $x \in M$ and $g, h \in G$.

For $\rho \in C^\infty(M \times G, M)$ and $g \in G$, we define a map $\rho^g\colon M \to M$ by $\rho^g(x) = \rho(x, g)$. Then ρ is an action if and only if the map $g \mapsto \rho^g$ is an anti-homomorphism from G into the group $\mathrm{Diff}^\infty(M)$ of diffeomorphisms of M. By $\mathcal{A}(M, G)$ we denote the subset of $C^\infty(M \times G, M)$ that consists of actions of G. It is a closed subspace of $C^\infty(M \times G, M)$. For $\rho \in \mathcal{A}(M, G)$ and $x \in M$, the set

$$\mathcal{O}_\rho(x) = \{\rho^g(x) \mid g \in G\}$$

is called the *ρ-orbit* of x.

Example 1.1.1. $\mathcal{A}(M, G)$ is non-empty for all M and G. In fact, it contains the *trivial action* ρ_{triv}, which is defined by $\rho_{\mathrm{triv}}(x, g) = x$. For every $x \in M$ we have that $\mathcal{O}_{\rho_{\mathrm{triv}}}(x) = \{x\}$.

Let us introduce an infinitesimal description of actions. By $\mathfrak{X}(M)$ we denote the Lie algebra of smooth vector fields on M. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ be the space of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{X}(M)$. In these notes, we identify $\mathfrak{g}$ with the subspace of $\mathfrak{X}(G)$ consisting of vector fields invariant under left translations. Each action $\rho \in \mathcal{A}(M, G)$ determines an associated *infinitesimal action* $I_\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$ by

$$I_\rho(\xi)(x) = \frac{d}{dt}\rho(x, \exp t\xi)\bigg|_{t=0}.$$

Proposition 1.1.2. *I_ρ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M)$.*

Proof. By $\mathcal{L}_X Y$ we denote the Lie derivative of a vector field Y with respect to (along) another vector field X. Take $\xi, \eta \in \mathfrak{g}$ and $x \in M$. Then,

$$\begin{aligned}
[I_\rho(\xi), I_\rho(\eta)](x) &= (\mathcal{L}_{I_\rho(\xi)} I_\rho(\eta))(x) \\
&= \lim_{t \to 0} \frac{1}{t} \left\{ D\rho^{\exp(-t\xi)}(I_\rho(\eta)(\rho^{\exp(t\xi)}(x))) - I_\rho(\eta)(x) \right\} \\
&= \frac{d}{dt} \left\{ \frac{d}{ds} (\rho^{\exp(-t\xi)} \circ \rho^{\exp(s\eta)} \circ \rho^{\exp(t\xi)})(x) \Big|_{s=0} \right\} \Big|_{t=0} \\
&= \frac{d}{dt} \left\{ \frac{d}{ds} \rho(x, \exp(t\xi)\exp(s\eta)\exp(-t\xi)) \Big|_{s=0} \right\} \Big|_{t=0} \\
&= \frac{d}{dt} \rho(x, \exp(t\mathrm{Ad}_{\exp(t\xi)}\eta)) \Big|_{t=0} \\
&= I_\rho([\xi, \eta])(x).
\end{aligned}$$

□

Proposition 1.1.3. *Two actions $\rho_1, \rho_2 \in \mathcal{A}(M, G)$ coincide if $I_{\rho_1} = I_{\rho_2}$. If G is simply connected and M is closed, then any $I \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ is the infinitesimal action associated with some action in $\mathcal{A}(M, G)$.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{A}(M, G)$. The curve $t \mapsto \rho_i(x, \exp(t\xi))$ is an integral curve of the vector field $I_{\rho_i}(\xi)$ for all $x \in M$, $\xi \in \mathfrak{g}$, and $i = 1, 2$. If $I_{\rho_1} = I_{\rho_2}$, then the uniqueness of integral curves implies that $\rho_1(x, \exp(t\xi)) = \rho_2(x, \exp(t\xi))$ for all $x \in M$, $t \in \mathbb{R}$, and $\xi \in \mathfrak{g}$. Since the union of one-parameter subgroups of G generates G, we have $\rho_1 = \rho_2$.

Suppose that G is simply connected and M is a closed manifold. Let E be the subbundle of $T(M \times G)$ given by

$$E(x, g) = \left\{ (I(\xi)(x), \xi(g)) \in T_{(x,g)}(M \times G) \mid \xi \in \mathfrak{g} \right\}.$$

For all $\xi, \xi' \in \mathfrak{g}$, we have

$$[(I(\xi), \xi), (I(\xi'), \xi')] = ([I(\xi), I(\xi')], [\xi, \xi']) = (I([\xi, \xi']), [\xi, \xi']).$$

By Frobenius' theorem, the subbundle E is integrable. Let $\mathcal{F}$ be the foliation on $M \times G$ generated by E. The space $M \times G$ admits a left action of G defined by $g \cdot (x, g') = (x, gg')$. The subbundle E is invariant under this action. Hence, we have $g \cdot \mathcal{F}(x, g') = \mathcal{F}(x, gg')$. Since M is compact, G is simply connected, and the foliation $\mathcal{F}$ is transverse to the natural fibration $\pi \colon M \times G \to G$, the restriction of π to each leaf of $\mathcal{F}$ is a diffeomorphism onto G. So, we can define a smooth map $\rho \colon M \times G \to M$ such that $\mathcal{F}(x, 1_G) \cap \pi^{-1}(g) = \{(\rho^g(x), g)\}$. Take $x \in M$ and $g, h \in G$. Then $(\rho^g \circ \rho^h(x), g)$ is contained in $\mathcal{F}(\rho^h(x), 1_G)$. Applying h from the left, we see that $(\rho^g \circ \rho^h(x), hg)$ is an element of $\mathcal{F}(\rho^h(x), h)$. Since $\mathcal{F}(\rho^h(x), h) = \mathcal{F}(x, 1_G)$ and $\{(\rho^{hg}(x), hg)\} = \mathcal{F}(x, 1_G) \cap \pi^{-1}(hg)$ by the definition of ρ, we have $\rho^g \circ \rho^h(x) = \rho^{hg}(x)$. Therefore, ρ is a right action of G. Now it is easy to check that $I_\rho = I$. □

We say that an action $\rho \in \mathcal{A}(M, G)$ is *locally free* if the isotropy group $\{g \in G \mid \rho^g(x) = x\}$ is a discrete subgroup of G for every $x \in M$. By $\mathcal{A}_{\mathrm{LF}}(M, G)$ we denote the set of locally free actions of G on M. Of course, the trivial action is not locally free unless M is zero-dimensional. The following is a list of basic examples of locally free actions.

Example 1.1.4 (Flows). A locally free $\mathbb{R}$-action is just a smooth flow with no stationary points. We remark that $\mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ is empty if M is a closed manifold with non-zero Euler characteristic.

Example 1.1.5 (The standard action). Let G be a Lie group, and Γ and H be closed subgroups of G. The *standard H-action* on $\Gamma\backslash G$ is the action $\rho_\Gamma \in \mathcal{A}(\Gamma\backslash G, H)$ defined by $\rho_\Gamma(\Gamma g, h) = \Gamma(gh)$. The action ρ is locally free if and only if $g^{-1}\Gamma g \cap H$ is a discrete subgroup of H for every $g \in G$. In particular, if Γ itself is a discrete subgroup of G, then ρ_Γ is locally free.

Example 1.1.6 (The suspension construction). Let M be a manifold and G be a Lie group. Take a discrete subgroup Γ of G, a closed subgroup H of G, and a left action $\sigma\colon \Gamma \times M \to M$. We put $M \times_\sigma G = M \times G/(x, g) \sim (\sigma(\gamma, x), \gamma g)$. Then $M \times_\sigma G$ is an M-bundle over $\Gamma\backslash G$. We define a locally free action ρ of H on $M \times_\sigma G$ by $\rho([x, g], h) = [x, gh]$.

We say that a homomorphism $I\colon \mathfrak{g} \to \mathfrak{X}(M)$ is *non-singular* if $I(\xi)(x) \neq 0$ for all $\xi \in \mathfrak{g} \setminus \{0\}$ and $x \in M$.

Proposition 1.1.7. *An action $\rho \in \mathcal{A}(M, G)$ is locally free if and only if I_ρ is non-singular.*

Corollary 1.1.8. *For any $\rho \in \mathcal{A}_{\mathrm{LF}}(M, G)$, the orbits of ρ form a smooth foliation. If the manifold M is closed, then the map $\rho(x, \cdot)\colon G \to \mathcal{O}(x, \rho)$ is a covering for any $x \in M$, where $\mathcal{O}(x, \rho)$ is endowed with the leaf topology.*

The proofs of the proposition and the corollary are easy and left to the reader. If M is closed, then the set of non-singular homomorphisms is an open subset of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$. Hence, $\mathcal{A}_{\mathrm{LF}}(M, G)$ is an open subset of $\mathcal{A}(M, G)$ in this case.

Let $\mathcal{F}$ be a foliation on a manifold M. We denote the tangent bundle of $\mathcal{F}$ by $T\mathcal{F}$ and the subalgebra of $\mathfrak{X}(M)$ consisting of vector fields tangent to $\mathcal{F}$ by $\mathfrak{X}(\mathcal{F})$. Let $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ be the set of locally free actions of a Lie group G whose orbit foliation is $\mathcal{F}$. The subspace $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ of $\mathcal{A}_{\mathrm{LF}}(M, G)$ is closed and consists of actions ρ such that I_ρ is an element of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(\mathcal{F}))$.

1.1.2 Rigidity and deformations of actions

We say that two actions $\rho_1 \in \mathcal{A}(M_1, G)$ and $\rho_2 \in \mathcal{A}(M_2, G)$ on manifolds M_1 and M_2 are (C^∞-)*conjugate* (and write $\rho_1 \simeq \rho_2$) if there exist a diffeomorphism $h\colon M_1 \to M_2$ and an automorphism Θ of G such that $\rho_2^{\Theta(g)} \circ h = h \circ \rho_1^g$ for every $g \in G$.

For a given foliation $\mathcal{F}$ on M, let $\mathrm{Diff}(\mathcal{F})$ be the set of diffeomorphisms of M which preserve each leaf of $\mathcal{F}$ and $\mathrm{Diff}_0(\mathcal{F})$ be its arc-wise connected component that contains Id_M. We say that two actions $\rho_1, \rho_2 \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ are (C^∞-) *parameter equivalent* (and write $\rho_1 \equiv \rho_2$) if they are conjugate by a pair (h, Θ) such that h is an element of $\mathrm{Diff}_0(\mathcal{F})$. It is easy to see that conjugacy and parameter equivalence are equivalence relations.

The ultimate goal of the study of smooth group actions is the classification of actions in $\mathcal{A}_{\mathrm{LF}}(M, G)$ or $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ up to conjugacy or parameter equivalence, for given G and M, or $\mathcal{F}$. The simplest case is that $\mathcal{A}_{\mathrm{LF}}(M, G)$ or $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ consists of only one equivalence class. We say that an action ρ_0 in $\mathcal{A}_{\mathrm{LF}}(M, G)$ is (C^∞-)*rigid* if any action in $\mathcal{A}_{\mathrm{LF}}(M, G)$ is conjugate to ρ_0. We say that an action ρ_0 whose orbit foliation is $\mathcal{F}$ is (C^∞-)*parameter rigid* if any action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ is parameter equivalent to ρ_0.

It is useful to introduce a local version of rigidity. We say that ρ_0 is *locally rigid* if there exists a neighborhood $\mathcal{U}$ of ρ_0 such that any action in $\mathcal{U}$ is conjugate to ρ_0. We also say that ρ_0 is *locally parameter rigid* if there exists a neighborhood $\mathcal{U}$ of ρ_0 in $\mathcal{A}(\mathcal{F}, G)$ such that any action in $\mathcal{U}$ is parameter equivalent to ρ_0. As we mentioned in the Introduction, local rigidity for compact group actions was settled in the early 1960's.

There exist actions which are locally parameter rigid, but not parameter rigid. For example, for $k \in \mathbb{Z}$, let ρ_k be the right action of $S^1 = \mathbb{R}/\mathbb{Z}$ on S^1 given by $\rho_k^t(s) = s + kt$. It is easy to see that ρ_1 is locally parameter rigid. Of course, all the orbits of ρ_k coincide with S^1 for $k \geq 1$. However, ρ_k is parameter equivalent to ρ_1 if and only if $|k| = 1$, since the mapping degree of $\rho_k(s, \cdot)$ is k. So, ρ_1 is locally parameter rigid, but not parameter rigid.

It is unknown whether every locally parameter rigid locally free action of a contractible Lie group on a closed manifold is parameter rigid or not.

Theorem 1.1.9 (Palais [44]). *Every action of a compact group on a closed manifold is locally rigid.*

As we will see later, many actions of non-compact groups fail to be locally rigid. Thus, it is natural to introduce the concept of deformation of actions. We say that a family $(\rho_\mu)_{\mu \in \Delta}$ of elements of $\mathcal{A}(M, G)$ parametrized by a manifold Δ is a C^∞ *family* if the map $\bar{\rho} : (x, g, \mu) \mapsto \rho_\mu(x, g)$ is smooth. By $\mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$ we denote the set of C^∞ families of actions in $\mathcal{A}_{\mathrm{LF}}(M, G)$ parametrized by Δ. Under the identification with $(\rho_\mu)_{\mu \in \Delta}$ and $\bar{\rho}$, the topology of $C^\infty(M \times G \times \Delta, M)$ induces a topology on $\mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$. We say that $(\rho_\mu)_{\mu \in \Delta}$ is a (*finite-dimensional*) *deformation* of $\rho \in \mathcal{A}(M, G)$ if Δ is an open neighborhood of 0 in a finite-dimensional vector space and $\rho_0 = \rho$.

In several cases, actions are not locally rigid, but their conjugacy class is *of finite codimension* in $\mathcal{A}_{\mathrm{LF}}(M, G)$. Here we formulate two types of finiteness of codimension. Let $(\rho_\mu)_{\mu \in \Delta} \in \mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$ be a deformation of ρ. We say that $(\rho_\mu)_{\mu \in \Delta}$ is *locally complete* if there exists a neighborhood $\mathcal{U}$ of ρ in $\mathcal{A}_{\mathrm{LF}}(M, G)$ such that any action in $\mathcal{U}$ is conjugate to ρ_μ for some $\mu \in \Delta$. We also say that $(\rho_\mu)_{\mu \in \Delta}$ is