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Open Problems in the Geometry and Analysis of Banach Spaces

Dedicated to the memory of
Joram Lindenstrauss,
Aleksander Pełczyński,
and Manuel Valdivia

Preface

This is a commented collection of some easily formulated open problems in the geometry and analysis on Banach spaces, focusing on basic linear geometry, convexity, approximation, optimization, differentiability, renormings, weak compact generating, Schauder bases and biorthogonal systems, fixed points, topology, and nonlinear geometry.

The collection consists of some commented questions that, to our best knowledge, are open. In some cases, we associate the problem with the person we first learned it from. We apologize if it may turn out that this person was not the original source. If we took the problem from a recent book, instead of referring to the author of the problem, we sometimes refer to that bibliographic source. We apologize that some problems might have already been solved. Some of the problems are long-standing open problems, some are recent, some are more important, and some are only "local" problems. Some would require new ideas, and some may go only with a subtle combination of known facts. All of them document the need for further research in this area. The list has of course been influenced by our limited knowledge of such a large field. The text bears no intentions to be systematic or exhaustive. In fact, big parts of important areas are missing: for example, many results in local theory of spaces (i.e., structures of finite-dimensional subspaces), more results in Haar measures and their relatives, etc. With each problem, we tried to provide some information where more on the particular problem can be found. We hope that the list may help in showing borders of the present knowledge in some parts of Banach space theory and thus be of some assistance in preparing MSc and PhD theses in this area. We are sure that the readers will have no difficulty to consider as well problems related to the ones presented here. We believe that this survey can especially help researchers that are outside the centers of Banach space theory. We have tried to choose such open problems that may attract readers' attention to areas surrounding them.

Summing up, the main purpose of this work is to help in convincing young researchers in functional analysis that the theory of Banach spaces is a fertile field of research, full of interesting open problems. Inside the Banach space area, the text should help a young researcher to choose his/her favorite part to work in. This way we hope that problems around the ones listed below may help in motivating research in these areas. For plenty of open problems, we refer also to, e.g., [AlKal06, BenLin00, BorVan10, CasGon97, DeGoZi93, Fa97, FHHMZ11, FMZ06, HaJo14, HMZ12, HMVZ08, HajZiz06, Kal08, LinTza77, MOTV09, Piet09, Woj91], and [Ziz03].

To assist the reader, we provided two indices and a comprehensive table referring to the listed problems by subject.

We follow the basic notation in the Banach space theory and assume that the reader is familiar with the very basic concepts and results in Banach spaces (see, e.g., [AlKal06, Di84, FHHMZ11, LinTza77, HMVZ08, Megg98]). For a very basic introduction to Banach space theory—"undergraduate style"—we refer to, e.g., [MZZ15, Chap. 11]. By a **Banach space** we usually mean an infinite-dimensional Banach space over the real field—otherwise we shall spell out that we deal with the finite-dimensional case. If no confusion may arise, the word **space** will refer to a Banach space. Unless stated otherwise, by a **subspace** we shall mean a closed subspace. The term **operator** refers to a bounded linear operator, and an operator with real values will be called a **functional**, understanding, except if it is explicitly mentioned, that it is continuous. A subspace Y of a Banach space X is said to be **complemented** if it is the range of a bounded linear projection on X. The **unit sphere** of the Banach space X, $\{x \in X : \|x\| = 1\}$, is denoted by S_X, and the **unit ball** $\{x \in X : \|x\| \leq 1\}$ is denoted by B_X. The words "smooth" and "differentiable" have the same meaning here. Unless stated otherwise, they are meant in the Fréchet (i.e., total differential) sense. If they are meant in the Gâteaux (i.e., directional) sense, we clearly mention it (for those concepts, see their definitions in, e.g., [FHHMZ11, p. 331]). We say that a norm is **smooth** when it is smooth at all nonzero points. Sometimes, we say that "a Banach space X admits a norm $\|\cdot\|$," meaning that it admits an *equivalent* norm $\|\cdot\|$. By **ZFC** we mean, as usual, the Zermelo-Fraenkel-Choice standard axioms of set theory. Unless stated otherwise, we use this set of axioms. We say that some statement is **consistent** if its negation cannot be proved by the sole ZFC set of axioms. Cardinal numbers are usually denoted by $\aleph$, while ordinal numbers are denoted by α, β, etc. With the symbol $\aleph_0$ we denote the cardinal number of the set $\mathbb{N}$ of natural numbers, and $\aleph_1$ is the first uncountable cardinal. Similarly, ω_0 (sometimes denoted ω) is the ordinal number of the set $\mathbb{N}$ under its natural ordering, and ω_1 is the first uncountable ordinal. The continuum hypothesis then reads $2^{\aleph_0} = \aleph_1$. The cardinality $2^{\aleph_0}$ of the set of real numbers (the **continuum**) will be denoted by c. If no confusion may arise, we sometimes denote by ω_1 also its cardinal number $\aleph_1$.

We prepared this little book as a working companion for [FHHMZ11] and [HMVZ08]. We often use this book to upgrade and update information provided in these two references.

Overall, we would be glad if this text helped in providing a picture of the present state of the art in this part of Banach space theory. We hope that the text may serve also as a kind of reference book for this area of research.

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The material comes from the interaction with many colleagues in meetings, in work, and in private conversations and, as the reader may appreciate in the comments to the problems, from many printed sources—papers, books, reviews, and even beamers from presentations—and, last, from our own research work. It is clear then that it will be impossible to explicitly thank so many influences. The authors prefer to carry on their own shoulders the responsibility for the selection of problems, eventual inaccuracies, wrong attributions, or lack of information about solutions. The names of authors appearing in problems, in comments, and in the reference list correspond to the panoply of mathematicians to whom thanks and acknowledgment usually appear in the introduction to a book.

Above all, the authors are indebted to their families for their moral support and encouragement.

The authors wish the readers a pleasant time spent over this little book.

Valencia, Spain — A.J. Guirao
Valencia, Spain — V. Montesinos
Calgary, Canada — V. Zizler
2016

Contents

Chapter 1
Basic Linear Structure

1.1 Schauder Bases

A sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space X is called a **Schauder basis for** X if for each $x \in X$ there is a unique sequence of scalars $\{\alpha_i\}_{i=1}^{\infty}$ such that $x = \sum_{i=1}^{\infty} \alpha_i e_i$. If the convergence of this series is **unconditional** for all $x \in X$ (i.e., any rearrangement of it converges), we say that the Schauder basis is **unconditional**. This is equivalent to say that under any permutation $\pi : \mathbb{N} \to \mathbb{N}$, the sequence $\{e_{\pi(i)}\}_{i=1}^{\infty}$ is again a basis of X.

Not every separable Banach space admits a Schauder basis,as it was shown first by P. Enflo (see, e.g., [LinTza77, p. 29] or [FHHMZ11, p. 711]).

We refer to [DLAT10, Pe06] and [Cass01] for more on the questions formulated in Problem 1.

Problem 1. Let X be a separable infinite-dimensional Banach space that is not isomorphic to a Hilbert space.

(i) (A. Pełczyński) Does there exist an infinite-dimensional subspace of X with Schauder basis that is not complemented in X?
(ii) (A. Pełczyński) Do there exist two infinite-dimensional subspaces of X with Schauder basis that are not isomorphic?
(iii) (A. Pełczyński) Does there exist a subspace of X with Schauder basis that is not isomorphic to ℓ_2?

(continued)

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Problem 1 (continued)
(iv) (G. Godefroy) Does there exist an infinite sequence $\{X_n\}_{n=1}^{\infty}$ of mutually nonisomorphic infinite-dimensional subspaces of X?
(v) Does there exist a subspace of X that has no unconditional Schauder basis?

Concerning Problem 1 we would like to point out:

1. J. Lindenstrauss and L. Tzafriri showed in [LinTza71] that *if all subspaces of a Banach space X are complemented in X, then X is isomorphic to a Hilbert space* (cf., e.g., [FHHMZ11, p. 309] or [AlKal06, p. 301]). On the other hand, W. B. Johnson and A. Szankowski in [JoSz14] showed that *there is a separable infinite-dimensional Banach space X that is not isomorphic to a Hilbert space and yet, every subspace of X is isomorphic to a complemented subspace of X.*
2. R. Komorowski and N. Tomczak-Jaegermann [KoTo95,98] and W. T. Gowers [Gow96] showed that *a separable Banach space X is isomorphic to a Hilbert space if all infinite-dimensional subspaces of X are isomorphic to X* (cf., e.g., [FHHMZ11, p. 267]).
3. W. B. Johnson showed that *there is a separable reflexive infinite-dimensional Banach space with unconditional Schauder basis* (the so-called **2-convexified Tsirelson space** T^2) *that does not contain isomorphic copies of* ℓ_p, $1 < p < \infty$, *and such that all of its subspaces do have Schauder basis* (cf., e.g., [Cass01, p. 276]).
4. A. Pełczyński and I. Singer showed in [PeSi64] that *if an infinite-dimensional Banach space X has a Schauder basis, then there is a continuum of normalized mutually non-equivalent conditional Schauder bases in X.* A Schauder basis $\{e_i\}_{i=1}^{\infty}$ is **normalized** if $\|e_i\| = 1$ for all i and a basis $\{e_i\}_{i=1}^{\infty}$ is **equivalent to a basis** $\{f_i\}_{i=1}^{\infty}$ if for scalars $\{\lambda_i\}_{i=1}^{\infty}$, $\sum \lambda_i e_i$ converges if and only if $\sum \lambda_i f_i$ converges.
5. R. Anisca *verified in* [An10] *in the positive G. Godefroy conjecture in Problem* 1 (iv) *for the class of the so-called weak Hilbert spaces that are not isomorphic to Hilbert spaces.* A Banach space X is a **weak Hilbert space** if there are positive constants K and δ such that every n-dimensional subspace of X has a subspace of dimension at least δn that is K-linearly isomorphic to a Hilbert space and K-complemented in X. The spaces X and Y are **K-linearly isomorphic** if there is a linear isomorphism φ from X onto Y such that $\|\varphi\| \cdot \|\varphi^{-1}\| \leq K$. The space Y is **K-complemented in** X if there is a projection P from X onto Y such that $\|P\| \leq K$. All subspaces, quotients, and duals of weak Hilbert spaces are themselves weak Hilbert spaces, and W. B. Johnson showed that *all weak Hilbert spaces are superreflexive* (for references see, e.g., [Pis88]). **Superreflexive spaces** are spaces that admit an equivalent uniformly convex norm. The norm $\|\cdot\|$ of a

Banach space is said to be **uniformly convex** if the **modulus of convexity** $\delta(\varepsilon) := \inf\{1 - \|(x+y)/2\| : x, y \in B_X, \|x-y\| \geq \varepsilon\}$ is positive for every $\varepsilon \in (0, 2]$.

6. A sequence in a normed space is said to be **basic** if it is a Schauder basis of its closed linear span. *Every infinite-dimensional Banach space contains a basic sequence* (a classic result of S. Mazur). W. T. Gowers and B. Maurey found, in 1991, *a reflexive Banach space X that has no unconditional basic sequence* (i.e., *X has no infinite-dimensional subspace with unconditional basis*). The result appeared in [GowMau93], and was inspired, as B. Maurey indicates in [Mau03], by the famous **B. S. Tsirelson example** of *a reflexive Banach space that does not contain any ℓ_p for $1 < p < +\infty$* [Tsi74] and by its modification by Th. Schlumprecht [Schl91]. It was crucial that W. B. Johnson showed that *this space of W. T. Gowers and B. Maurey is actually* ***hereditarily indecomposable*** (**HI**, in short), i.e., a Banach space X such that no closed subspace Z of X can be written as a topological direct sum of two infinite-dimensional closed subspaces of Z. This means that for every pair of two closed infinite-dimensional subspaces Y and Z of such X, the distance of S_Y to S_Z is zero. This in turn means that in such X, there is no bounded projection P from a subspace Z into itself such that the range of P and the kernel of P were infinite-dimensional (see also Sect. 1.2 below). Note that *all HI spaces clearly have the property that they do not contain any unconditional basic sequence*—since the span of such a sequence would be clearly decomposable. It follows that *if X is a hereditarily indecomposable space, then X is not isomorphic to any proper subspace of X; in particular, it is not isomorphic to any of its hyperplanes*. This was an open problem from S. Banach himself (cf., e.g., [Mau03, p. 1265]). The first example of a Banach space not isomorphic to its hyperplanes was found by W.T. Gowers in [Gow94]. This space has an unconditional basis. See also Problem 2 and Remarks to it, as well as Sect. 1.2 below.

   These results solved problems that have stayed open for about 70 years and created a true revolution in the recent development of Banach space theory. For this and more information we recommend to consult, e.g., [Mau03].

   Let us note in passing that *there is a nonseparable $C(K)$-space such that every one-to-one operator from $C(K)$ into itself is necessarily onto* [AviKo13].

   W. T. Gowers proved in [Gow96] the following dichotomy: *Let X be an arbitrary infinite-dimensional Banach space. Either X contains an unconditional basic sequence or X contains a HI subspace*. He also produced *a Banach space Y not containing any reflexive infinite-dimensional subspace and containing no copies of c_0 or ℓ_1* [Gow94b]. By James' theorem (cf., e.g., [FHHMZ11, p. 204]), together with the dichotomy just mentioned, we get a HI subspace of Y that has no reflexive subspace.

   In this direction see also [Fer97].

A **bump function** (or just a **bump**) on a Banach space is a real-valued function with bounded nonempty support. R. Deville showed that *if a Banach space X admits a C^∞-smooth bump function, then X contains a copy of c_0 or some ℓ_p, $p > 1$*, so it cannot be hereditarily indecomposable (see Remark 6 to Problem 1; see also [DeGoZi93, p. 209]).

We do not know the answer to the following problem:

Problem 2. Assume that X is a separable infinite-dimensional Banach space that admits a C^∞-smooth bump function. Is X necessarily isomorphic to its hyperplanes?

1. The very original space T of Tsirelson (see Remark 6 to Problem 1 above) was constructed as *a reflexive space with unconditional basis, no infinite-dimensional subspace of which admits a uniformly convex norm.* This short self-contained crystal-clear text has drastically influenced the whole Banach space theory. This was achieved by ensuring that *for every infinite-dimensional subspace E of T, c_0 is* ***crudely finitely representable in*** *E* (meaning that there is $K > 0$ such that every finite-dimensional subspace of c_0 is K-isomorphic to a subspace of E). So, c_0 is crudely finitely representable in each infinite-dimensional subspace of T and yet, c_0 is not isomorphic to any subspace of T (T is reflexive). Now, if an infinite-dimensional subspace E of T had an equivalent uniformly convex norm, by a simple limit technique explained, e.g., in [FHHMZ11, p. 435], this would give that c_0 admits an equivalent uniformly convex norm. Since c_0 does not admit any uniformly convex norm as it is not reflexive [FHHMZ11, p. 434], this all implies that no infinite-dimensional subspace of T can have an equivalent uniformly convex norm. Thus, in particular, no ℓ_p for $p > 1$ can be isomorphic to a subspace of T. As a reflexive space, T cannot contain an isomorphic copy of ℓ_1. Therefore T cannot contain a copy of any ℓ_p or c_0. Tsirelson's original, truly ingenious, short, direct geometric construction of the unit ball of T [Tsi74] is described, e.g., in [FHHMZ11, p. 459]. The key point is the construction of the unit ball of T as a weakly compact subset of c_0, in such a way that, by Pełczyński method, one can model finitely c_0 on the "tails" of sequences. This kind of modelling is the main novelty in Tsirelson construction. It is proved in [CJT84] that *T isomorphically embeds into each infinite-dimensional subspace of T*. An analytic approach to the Tsirelson (dual) space is explained in [LinTza77, p. 95]. This space thus solved the original Banach problem on containment of ℓ_p or c_0 in every Banach space which used to be a famous longstanding problem for about 40 years. Tsirelson's example had an enormous impact on the Banach space theory and has been substantially influencing its further development since the year 1974, when it appeared. The reader is encouraged to consult [CaSh89].

   As a reflexive space, Tsirelson's space admits a Fréchet differentiable norm (cf., e.g., [FHHMZ11, p. 387]). However, the (continuous) differential of this norm cannot be locally uniformly continuous on (the sphere of) any infinite-dimensional subspace of the Tsirelson space (cf., e.g., [DeGoZi93, p. 203]).

A space introduced by T. Figiel and W. B. Johnson in [FiJo74] can be considered as the first descendent of Tsirelson's space. *It is a uniformly convex space that contains no copies of* ℓ_p *or* c_0. Then, after, say, 15 years, T. Schlumprecht constructed a more flexible variant of Tsirelson's space—presented in the Jerusalem conference in 1991 and now called the **space** S, see [Schl91]—that almost immediately created a true revolution in Banach space theory, leading to the solution of the hyperplane problem, the unconditional subbasis problem, the homogeneous problem and, above all, the creation of a HI space (as we discussed in Comments to Problem 1; see also Sect. 1.2 below). Moreover it led to results on distortable norms.

2. The first example of a **Fréchet space** (i.e., a locally convex complete metric linear space) with the property of not being isomorphic to a subspace of codimension 1 was constructed by C. Bessaga, A. Pełczyński, and S. Rolewicz in 1961 in [BePeRo61]. E. Dubinsky then proved that, in particular, *every separable Banach space has a dense subspace that is not isomorphic to its hyperplanes* [Dub71]. We refer to [PeBe79, p. 227] for more on this subject.

   The fact that it took half of a century quite an effort of many world centers to do such construction for Banach spaces documents how subtle and creative the concept of Banach space is.

The **basis constant** $\mathrm{bc}(Y)$ of a Banach space Y is defined as the least upper bound of the constants L such that there is a Schauder basis $\{e_n\}_{n=1}^{\infty}$ for Y satisfying

$$\Big\|\sum_{j=1}^{n} t_j e_j\Big\| \leq L \Big\|\sum_{j=1}^{n+m} t_j e_j\Big\| \text{ for all scalars } t_1, \dots t_{n+m}, \text{ and } n, m \in \mathbb{N}.$$

If Y does not have any basis we put $\mathrm{bc}(Y) = \infty$.

If X and Y are two isomorphic Banach spaces, then the **Banach–Mazur distance** between X and Y is defined to be the infimum of $\|T\|.\|T^{-1}\|$ as T ranges over all isomorphisms from X onto Y. The Banach–Mazur distance between X and Y is denoted by $d(X, Y)$.

Parts of Problem 1 are closely connected to the following more general conjecture of A. Pełczyński [Pe06].

Problem 3 (A. Pełczyński). Does there exist a constant $C \geq 1$ and a function $\varphi : [1, +\infty) \to \mathbb{R}$ with $\lim_{L\to\infty} \varphi(L) = \infty$ such that if $\dim E < \infty$ and $d(E, \ell_2^{\dim E}) \geq L$ then there is a subspace $F \subset E$ with $\mathrm{bc}(F) \leq C$ and $d(F, \ell_2^{\dim F}) \geq \varphi(L)$?

Roughly speaking the conjecture says: *A finite-dimensional space which is far from a Hilbert space has a subspace which is far from a Hilbert space and has a nice basis.*

We do not know if the following problem from [LinTza77, p. 86] is still open:

Problem 4. Let $1 < p < \infty$ and let X be an infinite-dimensional Banach space that is isomorphic both to a subspace and to a quotient space of ℓ_p. Is X isomorphic to ℓ_p?

Recall that the **density of a Banach space** X is the minimal cardinality of a norm dense set in X.

Problem 5. Can the original Tsirelson's construction mentioned in Remark 6 to Problem 1, above, be adjusted to produce a reflexive Banach space X of density c with unconditional basis such that c_0 is crudely finitely representable in every infinite-dimensional subspace of X?

We defined unconditional Schauder basis at p. 1. In general, a family $\{e_\gamma\}_{\gamma\in\Gamma}$ of vectors in a Banach space X is called an **unconditional long Schauder basis** of X if for every $x \in X$ there is a unique family of real numbers $\{a_\gamma\}_{\gamma\in\Gamma}$ such that $x = \sum a_\gamma e_\gamma$ in the sense that for every $\varepsilon > 0$ there is a finite set $F \subset \Gamma$ such that $\|x - \sum_{\gamma\in F'} a_\gamma e_\gamma\| \leq \varepsilon$ for every finite $F' \supset F$.

The following problem is mentioned, e.g., in [LinTza73, p. 19].

Problem 6. Assume that X is a separable Banach with unconditional Schauder basis and Y is a complemented subspace of X. Does Y have an unconditional Schauder basis? Or, at least, does such Y have a complemented subspace with an unconditional Schauder basis?

We refer to [Cass01, p. 279].

A Banach space X is called an $\mathcal{L}_p$**-space**, for $1 \leq p \leq \infty$, if there is $\lambda < \infty$ such that for every finite-dimensional subspace E of X, there is a further finite-dimensional subspace F of X with $F \supset E$ and with $d(F, \ell_p^{\dim F}) \leq \lambda$.

Problem 7. Assume that $1 < p < \infty$, $p \neq 2$, and that X is a separable $\mathcal{L}_p$-space. Does X have an unconditional Schauder basis?

Note that *X is then a complemented subspace of L_p* [JoLin01b, p. 57]. Thus the problem is connected with Problem 6 (use the Marcinkiewicz–Paley theorem on unconditional bases in L_p spaces for $p > 1$, see, e.g., [AlKal06, p. 130]).

This problem is in [HOS11], where more on it can be found; for example, that *any separable $\mathcal{L}_p$-space has a Schauder basis for $1 \leq p < \infty$.*

Problem 8. Let $1 < p < \infty$, $p \neq 2$. Assume that $L_p(\mu)$ has density $\aleph_1$ and μ is finite. Does $L_p(\mu)$ have an unconditional Schauder basis?

We took this problem from [JoSch14]. It is known that *the answer is negative if the density of the space is at least $\aleph_\omega$* [EnRo73].

Connected with Problem 8, we may ask the following:

Problem 9. Study the long Schauder bases in $L_p(\mu)$-spaces for μ finite in the sense described, e.g., in [HMVZ08, p. 132].