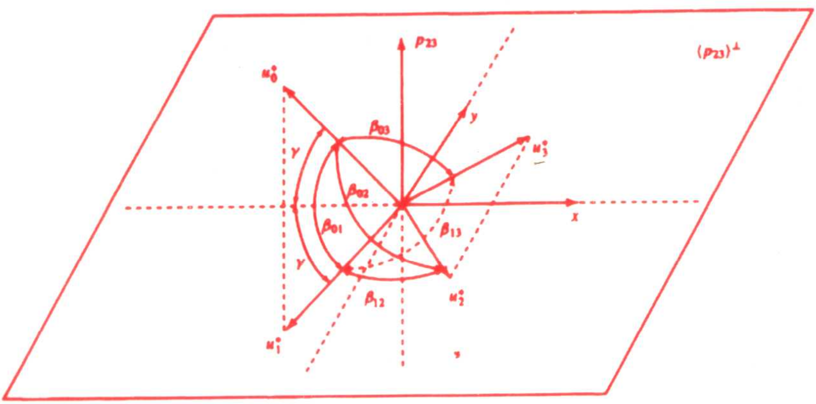
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 61

GEOMETRIC APPLICATIONS OF FOURIER SERIES AND SPHERICAL HARMONICS

H. GROEMER

傅立叶级数和球面调和函数的几何应用



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Geometric Applications of Fourier Series and Spherical Harmonics

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书　　名：Geometric Applications of Fourier Series and Spherical Harmonics
作　　者：H.Groemer
中 译 名：傅立叶级数和球面调和函数的几何应用
出 版 者：世界图书出版公司北京公司
印 刷 者：北京中西印刷厂
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）
开　　本：1/24　711×1245　印　张：14.5
出版年代：2000 年 6 月
书　　号：ISBN 7-5062-4701-1/ O・284
版权登记：图字 01-2000-0860
定　　价：58.00 元

PREFACE

In 1901 Adolf Hurwitz published a short note showing that Fourier series can be used to prove the isoperimetric inequality for domains in the Euclidean plane, and in a subsequent article he showed how spherical harmonics can be utilized to prove an analogous inequality for three-dimensional convex bodies. A few years later Hermann Minkowski used spherical harmonics to prove an interesting characterization of (three-dimensional) convex bodies of constant width. The work of Hurwitz and Minkowski has convincingly shown that a study of this interplay of analysis and geometry, in particular of Fourier series and spherical harmonics on the one hand, and the theory of convex bodies on the other hand, can lead to interesting geometric results. Since then many articles have appeared that explored the possibilities of such methods.

The aim of the present book is to provide a fairly comprehensive exposition of geometric results, more specifically, of results in the theory of convex sets, that have been proved by the use of Fourier series or spherical harmonics. Almost all theorems that are stated are also proved. Furthermore, to make the book more self-contained, all results from the theory of spherical harmonics that are used are also proved. Thus the only prerequisite for reading this book is some familiarity with the basic facts of the theory of (finite dimensional) convex sets and the theory of functions of real variables.

The book consists of five chapters. The first two of these contain preparatory material from analysis and the geometry of convex sets. These topics are reviewed to establish consistent notation and to formulate some known results for later reference. Moreover, some auxiliary results that are not part of the standard textbook literature are formulated and proved.

In Chapter 3 the theory of spherical harmonics is developed to the extent that it is useful for later geometric applications. An attempt has been made to present this material at the level of classical analysis, since a more abstract approach would not have significantly enlarged the area of geometric applications, but could have made it more difficult for some readers to acquaint themselves with the necessary analytic tools.

Chapters 4 and 5 contain the geometric applications. Readers familiar with the theory of convex sets and spherical harmonics may start immediately with these chapters, using the previous material only to acquaint themselves with the notation, definitions, and some auxiliary results. Chapter 4 deals with applications of Fourier series and consequently with geometric results in the two-dimensional Euclidean space. Some of these results are discussed since higher dimensional analogues are not available, others because they provide good examples of the method of proof that will be used later in the more complicated higher dimensional situation. Chapter 5 can be considered the principal part of the book. It deals with applications of spherical harmonics to geometric problems in Euclidean space without dimension limitations.

Some of the results presented here are proved in essentially the same way as in the articles where they originally appeared; in other cases the proofs have been modified considerably or even replaced by new ones if this was deemed to benefit the overall presentation of the subject. Several theorems too far removed from the central area of this book or better proved by other methods are mentioned in a more casual manner. This is usually done at the end of each section in a kind of appendix headed "Remarks and References." In these paragraphs one also finds the pertinent references to the original literature and some historical comments. No attempt has been made, however, to present a comprehensive survey of the long and ramified history of spherical harmonics and their applications.

Several mathematicians were kind enough to read at least some parts of earlier versions of this book and to make valuable comments and suggestions. In particular, I wish to thank Professors Gulbank D. Chakerian, Paul R. Goodey, Richard J. Gardner, Peter M. Gruber, Erwin Lutwak, Krzysztof Przesławski, and Rolf Schneider. I would like to express my special gratitude to Professor Erwin Lutwak, who originally suggested that I write a book for the Encyclopedia series, and to Professor Rolf Schneider, whose extensive contributions and interesting results in the subject area of this book have stimulated me to study this topic and to write about it.

Remarks on Matters of Presentation and Notation

Definitions are stated in the conventional way with "if" instead of "if and only if." The concepts of lemmas, theorems, and corollaries are used with their well-established meanings, but there also appear "propositions." These are meant to be either auxiliary results having some independent interest or significance, or results of a less important or more isolated kind than those listed as theorems.

All lemmas, theorems, corollaries, propositions, and some formulas are marked by three numbers. The first one indicates the chapter, the second number refers to the section within the chapter, and the third one indicates the position within the section. References are given by the name of the author (or the names of the authors) in small capital letters followed by the year of publication, sometimes marked with a letter if there are several articles by the same author within one year. Depending on the context, when such a reference is given it may mean either the author or the publication. For example a phrase like "see MINKOWSKI (1903)" means obviously that one should see the article which Minkowski published in 1903 and which is listed as such in the References at the end of the book. On the other hand a statement like "this was proved by MINKOWSKI (1903)" clearly refers to the person and the year when the result was published. If an article or a book has more than one publication date the earliest one is used for reference; if it has not yet appeared in print it is quoted as "in press," possibly followed by a letter if there is more than one such prospective publication by the same author.

The word "function" without any further information on its range is always meant to be a real valued function. Similarly, the noun "constant" without any further specification is used to indicate a real valued constant. We sometimes employ such convenient and customary (but mildly contradictory) phrases as "c_n is a constant depending on n only," which, strictly speaking, means that c_n is a function of the single variable n.

There is a list of frequently used symbols at the end of the book (after the References). Of particular importance is that, except for its use in integrals and derivatives, the letter d will always denote the dimension of the space under consideration. Unless specified otherwise, it is assumed that $d \geq 2$. It also should be observed that κ_d and σ_d are used exclusively to denote, respectively, the volume and surface area of a d-dimensional unit ball. The term "measure" always means a nonnegative measure; if negative values are permitted the corresponding concept will be called a signed measure.

CONTENTS

1

Analytic Preparations

We review here some of the analytic concepts and facts that will be used in later chapters. Most of this material forms part of the standard textbook literature on real analysis or functional analysis and it is not necessary to repeat here the pertinent proofs. However, a few facts of a more special character and not generally known will be formulated as lemmas and proved.

Throughout this book we let $\mathbf{E}^d$ denote the Euclidean d-dimensional space. If x is a point of $\mathbf{E}^d$ the coordinates of x will be denoted by x_i; hence, $x = (x_1, \ldots, x_d)$. The letter o denotes the origin $(0, \ldots, 0)$ of $\mathbf{E}^d$. If $u, v \in \mathbf{E}^d$ we let $u \cdot v$ denote the inner product, and $|u|$ the Euclidean norm. Of course, for points in $\mathbf{E}^1$, that is, for real numbers, $|\cdot|$ is the ordinary absolute value. The Lebesgue measure of a subset S of $\mathbf{E}^d$ will usually be called the *volume* of S and denoted by $v(S)$. We write $B^d(p, r)$ for the closed ball in $\mathbf{E}^d$ of radius r centered at p, and $B^d = B^d(o, 1)$ for the closed unit ball in $\mathbf{E}^d$ centered at o. Furthermore, we let S^{d-1} denote the boundary of B^d, that is, the unit sphere in $\mathbf{E}^d$. The spherical Lebesgue measure on S^{d-1} will be denoted by σ, the volume of B^d by κ_d, and the surface area of B^d by σ_d.

1.1. Inner Product, Norm, and Orthogonality of Functions

We let $L(S^{d-1})$ denote the class of integrable functions on S^{d-1}, and $L_2(S^{d-1})$ the class of square integrable functions on S^{d-1}. Thus, $L_2(S^{d-1})$ consists of all real valued Lebesgue integrable functions F on S^{d-1} with the property that

$$\int_{S^{d-1}} F(u)^2 d\sigma(u) < \infty.$$

Of course in such definitions the underlying measure space is always $(S^{d-1}, \mathcal{M}, \sigma)$ with $\mathcal{M}$ denoting the class of subsets of S^{d-1} that are measurable with respect to the spherical Lebesgue measure σ. If $F, G \in L(S^{d-1})$ the *inner product* $\langle F, G\rangle$ is defined by

$$\langle F, G\rangle = \int_{S^{d-1}} F(u)G(u)d\sigma(u)$$

(provided that this integral exists). We let $\|\cdot\|$ denote the norm derived from this inner product. Hence, $\|F\| = \langle F, F\rangle^{1/2}$, and if $F, G \in L_2(S^{d-1})$, then the triangle inequality $\|F+G\| \leq \|F\| + \|G\|$ and the Cauchy–Schwarz inequality $\langle F, G\rangle \leq \|F\|\|G\|$ are valid. It is well-known that with respect to ordinary addition and scalar multiplication, and with the usual identification of functions that are equal almost everywhere, $L_2(S^{d-1})$ is a Hilbert space. More generally, if Φ is a function on S^{d-1} with values in $\mathbf{E}^d$ then $\|\Phi\|$ is defined as the functional norm of the Euclidean norm of Φ. In other words,

$$\|\Phi\| = \left(\int_{S^{d-1}} |\Phi(u)|^2 d\sigma(u)\right)^{1/2}.$$

If $F, F_i \in L_2(S^{d-1})$ $(i = 0, 1, \ldots)$ and

$$\lim_{i\to\infty} \|F_i - F\| = 0$$

the sequence $F_0, F_1, \ldots$ will be said to *converge in mean* to F. Of course, an infinite series of functions from $L_2(S^{d-1})$ is said to converge in mean to some function $G \in L_2(S^{d-1})$ if the sequence of the corresponding partial sums converges in mean to G. Of particular importance in this connection is the fact that $L_2(S^{d-1})$, being a Hilbert space, is complete in the sense that every Cauchy sequence (with respect to the norm in $L_2(S^{d-1})$) is (in mean) convergent.

Two functions F, G from $L_2(S^{d-1})$ are said to be *orthogonal* if $\langle F, G\rangle = 0$. A sequence $H_0, H_1, \ldots$ with $H_i \in L_2(S^{d-1})$ and $\|H_i\| \neq 0$ (for all i) will be called an *orthogonal sequence* if $\langle H_i, H_j\rangle = 0$ whenever $i \neq j$. It is called an *orthonormal sequence* if $\langle H_i, H_j\rangle = \delta_{ij}$, where, as usual,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = k. \end{cases}$$

If $F \in L_2(S^{d-1})$ and $H_0, H_1, \ldots$ is a given orthogonal sequence, then the numbers

$$\alpha_i = \frac{\langle F, H_i\rangle}{\|H_i\|^2}$$

are called the *Fourier coefficients* of F (with respect to the given orthogonal sequence), and the series

$$\sum_{i=0}^{\infty} \alpha_i H_i$$

is called the *Fourier series* of F (with respect to the sequence $H_0, H_1, \ldots$). To indicate that $\sum_{i=0}^{\infty} \alpha_i H_i$ is the Fourier series of a given function F we write

$$F \sim \sum_{i=0}^{\infty} \alpha_i H_i. \tag{1.1.1}$$

If a and b are real numbers, and if, in addition to (1.1.1), $G \in L_2(S^{d-1})$ and

$$G \sim \sum_{i=0}^{\infty} \beta_i H_i,$$

then

$$aF + bG \sim \sum_{i=0}^{\infty} (\alpha_i H_i + \beta_i G_i).$$

From the definition of the Fourier coefficients it follows immediately that

$$\left\| F - \sum_{i=0}^{m} \alpha_i H_i \right\|^2 = \|F\|^2 - \sum_{i=0}^{m} \alpha_i^2 \|H_i\|^2. \tag{1.1.2}$$

An obvious consequence of this is *Bessel's inequality*

$$\sum_{i=0}^{\infty} \alpha_i^2 \|H_i\|^2 \leq \|F\|^2.$$

Another consequence of (1.1.2) is the fact that the equality

$$\lim_{m\to\infty} \left\| F - \sum_{i=0}^{m} \alpha_i H_i \right\| = 0 \tag{1.1.3}$$

holds if and only if

$$\|F\|^2 = \sum_{i=0}^{\infty} \alpha_i^2 \|H_i\|^2. \tag{1.1.4}$$

The latter relation is called *Parseval's* equation. Thus one can state that the Fourier series of a function $F \in L_2(S^{d-1})$ with respect to a given orthogonal sequence $H_0, H_1, \ldots$ of functions from $L_2(S^{d-1})$ converges in mean to F if and only if Parseval's equation (1.1.4) holds. Furthermore, it is not difficult to prove that (1.1.3) (or, equivalently, (1.1.4)) holds for all $F \in L_2(S^{d-1})$ if and only if the given orthogonal sequence $H_0, H_1, \ldots$ has the following property: Whenever $G \in L_2(S^{d-1})$ and $\langle G, H_i \rangle = 0$ (for all i), then $G = 0$ almost everywhere. An orthogonal sequence $H_1, H_2, \ldots$ with the property that for every $F \in L_2(S^{d-1})$

the relation (1.1.3) holds is said to be *complete* or (to avoid confusion with the completeness of $L_2(S^{d-1})$) *total*. If a sequence $H_0, H_1, \ldots$ is complete and $F \sim \sum_{i=0}^{\infty} \alpha_i H_i$, $G \sim \sum_{i=0}^{\infty} \beta_i G_i$ it follows from (1.1.4) applied to F, G, and $F + G$, that

$$\langle F, G \rangle = \sum_{i=0}^{\infty} \alpha_i \beta_i \|H_i\|^2. \tag{1.1.5}$$

This relation is called the *generalized Parseval's equation*. An immediate consequence of Parseval's equation (1.1.4), applied to $F - G$, is that the assumptions $F, G \in L_2(S^{d-1})$ and

$$F \sim \sum_{i=0}^{\infty} \alpha_i H_i, \qquad G \sim \sum_{i=0}^{\infty} \alpha_i H_i$$

imply

$$F = G \quad \text{a.e.}$$

(As usual, the abbreviation "a.e." means "almost everywhere.") Another consequence worth mentioning is the fact that if, for some integer m,

$$F \sim \sum_{i=0}^{m} \alpha_i H_i,$$

then

$$F = \sum_{i=0}^{m} \alpha_i H_i \quad \text{a.e.}$$

We finally state a lemma that expresses a well-known approximation property of the Fourier coefficients.

Lemma 1.1.1. *Let $F, F_1, \ldots, F_n$ be mutually orthogonal functions belonging to $L_2(S^{d-1})$ and assume that $\|F_i\| \neq 0$ (for all i). Then $\|F - \sum_{i=1}^{n} \gamma_i F_i\|$, considered as a function of $\gamma_1, \ldots, \gamma_n$, is minimal if and only if $\gamma_i = \langle F, F_i \rangle / \|F_i\|^2$.*

Proof. This lemma follows immediately from the easily proved relation

$$\left\| F - \sum_{i=0}^{n} \gamma_i F_i \right\|^2 = \left\| F - \sum_{i=1}^{n} \alpha_i F_i \right\|^2 + \sum_{i=1}^{n} (\alpha_i - \gamma_i)^2 \|F_i\|^2,$$

where $\alpha_i = \langle F, F_i \rangle / \|F_i\|^2$. □

Occasionally it will be convenient to use the supremum norm $\|\cdot\|_\infty$ on S^{d-1}. If F is a real valued continuous function on S^{d-1}, or a continuous function mapping S^{d-1} into $\mathbf{E}^d$, it can be defined by

$$\|F\|_\infty = \max\{|F(u)| : u \in S^{d-1}\},$$

where $|\cdot|$ denotes, respectively, either the absolute value or the Euclidean norm.

Remarks and References. The concepts of inner product, norm, orthogonality, etc., that are based on the Hilbert space structure of $L_2(S^{d-1})$ are developed (often in a much more general setting) in most textbooks of real analysis or functional analysis, for example, in RUDIN (1974) or ROYDEN (1988). In Section 3.3, we introduce a slightly modified concept of orthogonality (on $[-1, 1]$) with respect to a weight function, but in the present book such variants will play only a minor and rather isolated role.

1.2. The Gradient and Beltrami Operator

If f is a function whose domain is a subset of $\mathbf{E}^d$ that contains S^{d-1} and whose range is in the set of real numbers or in $\mathbf{E}^d$, we write $\hat{f}$ or $f^\wedge$ for the restriction of f to S^{d-1}. If, on the other hand, F is defined on S^{d-1} we let $\check{F}$ or $F^\vee$ denote the *radial extension* of F to $\mathbf{E}^d/\{o\}$. This means that

$$\check{F}(x) = F(x/|x|).$$

Note that always $(F^\vee)^\wedge = F$ whereas in general $(f^\wedge)^\vee \neq f$.

A function F on S^{d-1} will be said to be n times differentiable (or n times continuously differentiable) if the partial derivatives of $\check{F}$ of order n exist (or exist and are continuous). For geometric applications of spherical harmonics one frequently has to work with the following second order differential operators. The Laplace operator Δ and the gradient ∇ are defined, respectively, by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

and

$$\nabla = e_1\frac{\partial}{\partial x_1} + \cdots + e_d\frac{\partial}{\partial x_d},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots 0) \in \mathbf{E}^d$ with 1 occurring at the ith position. Both Δ and ∇ operate on any sufficiently smooth function defined on an open subset of $\mathbf{E}^d$.

Using the above extension procedure one can transfer both the Laplace operator and the gradient to operators acting on functions on S^{d-1}. These operators will be denoted, respectively, by ∇_o and Δ_o and are defined by

$$\Delta_o F = (\Delta \check{F})^\wedge$$

and

$$\nabla_o F = (\nabla \check{F})^\wedge.$$

So $\Delta_o F$ and $\nabla_o F$ exist if F is, respectively, twice or once differentiable. The operator Δ_o is usually called the *Laplace–Beltrami operator* or simply the *Beltrami operator*, whereas ∇_o is again referred to as the *gradient*.

On several occasions we will need *Green's formula*. If Q is the closure of an open set in $\mathbf{E}^d$ with sufficiently smooth boundary and if f and g are twice continuously differentiable functions on an open set containing Q, then this formula can be stated as

$$\int_{\partial Q} f D_q g ds = \int_Q ((\nabla f) \cdot (\nabla g) + f \Delta g) dv, \tag{1.2.1}$$

where dv denotes the volume differential, ds the surface area differential, and D_q the directional derivative in the outer normal direction q of ∂Q. Actually only the special cases when Q is a ball or the set between two concentric balls will be needed. In particular, if $Q = B^d$ (1.2.1) can be written in the form

$$\int_{S^{d-1}} f D_q g d\sigma = \int_{B^d} ((\nabla f) \cdot (\nabla g) + f \Delta g) dv. \tag{1.2.2}$$

Interchanging f and g in (1.2.2) and forming the difference one obtains the following relation, which is also often referred to as Green's formula:

$$\int_{S^{d-1}} (f D_q g - g D_q f) d\sigma = \int_{B^d} (f \Delta g - g \Delta f) dv. \tag{1.2.3}$$

As another special case of (1.2.1) consider two twice continuously differentiable functions F, G on S^{d-1}, and let $f = \check{F}$, $g = \check{G}$. If $r_o \in (0, 1)$ and Q is the closure of $B^d(o, r) \backslash B^d(o, r_o)$, where $r > r_o$, then, noting that f and g are constant in the radial direction, we have $D_q f = D_q g = 0$ and it follows from (1.2.1) that

$$\int_{r_o}^{r} \left(\int_{\partial B^d(o,\rho)} ((\nabla f(x)) \cdot (\nabla g(x)) + f(x) \Delta g(x)) d\sigma(\rho, x) \right) d\rho = 0,$$

where $d\sigma(\rho, x)$ denotes the surface area differential on $\partial B^d(o, \rho)$. Differentiating with respect to r and subsequently letting $r = 1$ one deduces that

$$\int_{S^{d-1}} ((\nabla f(x)) \cdot (\nabla g(x)) + f(x) \Delta g(x)) d\sigma(x) = 0.$$

In view of the respective definitions of f, g, Δ_o, and ∇_o this can be written in the form

$$\int_{S^{d-1}} F\Delta_o G d\sigma = -\int_{S^{d-1}} (\nabla_o F)\cdot(\nabla_o G) d\sigma. \tag{1.2.4}$$

As an immediate consequence of (1.2.4) one can state that

$$\int_{S^{d-1}} F\Delta_o G d\sigma = \int_{S^{d-1}} G\Delta_o F d\sigma \tag{1.2.5}$$

and, if $F = G$,

$$\int_{S^{d-1}} F\Delta_o F d\sigma = -\int_{S^{d-1}} |\nabla_o F|^2 d\sigma. \tag{1.2.6}$$

The equalities (1.2.5) and (1.2.6) can be formulated more concisely as

$$\langle F, \Delta_o G\rangle = \langle \Delta_o F, G\rangle \tag{1.2.7}$$

and

$$\langle F, \Delta_o F\rangle = -\|\nabla_o F\|^2. \tag{1.2.8}$$

The fact that (1.2.7) holds can also be expressed by stating that Δ_o is a self-adjoint operator.

Another important relation is obtained by assuming that f is a twice differentiable function on $\mathbf{E}^d\setminus\{o\}$ that is positively homogeneous of degree m (that means, if $t > 0$, then $f(tx) = t^m f(x)$). Using Euler's relation $\sum_{i=1}^{d} x_i \frac{\partial f(x)}{\partial x_i} = mf(x)$ one finds after a straightforward calculation that $\Delta f(x/|x|) = \Delta(f(x)|x|^{-m}) = |x|^{-m}\Delta f(x) - m(m+d-2)f(x)$. Hence, if $u \in S^{d-1}$, then

$$(\Delta_o \hat{f})(u) = (\Delta f)(u) - m(m+d-2)f(u). \tag{1.2.9}$$

A similar, but even easier calculation shows that for any function f on $\mathbf{E}^d\setminus\{o\}$ that is differentiable and positively homogeneous of degree 1 we have

$$(\nabla_o \hat{f})(u) = (\nabla f)(u) - f(u)u. \tag{1.2.10}$$

In the case $d = 2$ it is often advantageous to work with polar coordinates. If G is a differentiable function on S^1 and $x_1 = r\cos\omega, x_2 = r\sin\omega$, then G can be viewed as a function of ω and $\check{G}(x_1, x_2)$ does not depend on r. Hence $\check{G}(x_1, x_2) = G(\omega)$ and $\partial\check{G}/\partial r = 0$. It follows that $dG/d\omega$ exists and

$$\frac{\partial\check{G}}{\partial x_1} = -\frac{1}{r}\frac{dG}{d\omega}\sin\omega, \qquad \frac{\partial\check{G}}{\partial x_2} = \frac{1}{r}\frac{dG}{d\omega}\cos\omega.$$

Consequently

$$\nabla_o G = \frac{dG}{d\omega}(-\sin\omega, \cos\omega) \tag{1.2.11}$$

and

$$\frac{dG}{d\omega} = (\nabla_o G) \cdot (-\sin\omega, \cos\omega). \tag{1.2.12}$$

An immediate consequence of this relation is that

$$|\nabla_o G| = \left|\frac{dG}{d\omega}\right| \tag{1.2.13}$$

(with the Euclidean norm on the left-hand side and the absolute value on the right-hand side). Concerning the Beltrami operator it is easily shown that if $G(x_1, x_2)$ is twice differentiable, then $d^2G/d\omega^2$ exists, and one can use the two-dimensional Laplace operator in polar coordinates, namely

$$\Delta\check{G} = \frac{\partial^2\check{G}}{\partial r^2} + \frac{1}{r}\frac{\partial\check{G}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\check{G}}{\partial\omega^2},$$

and the fact that $\partial\check{G}/\partial r = \partial^2\check{G}/\partial r^2 = 0$ to conclude that

$$\Delta_o G = \frac{d^2G}{d\omega^2}.$$

Remarks and References. The Laplace–Beltrami operator is an often used concept in differential geometry. It can be defined for more general manifolds than spheres. However such generalizations, which are of importance in differential geometry, are irrelevant for our applications. The above introduction of this operator follows essentially that of SEELEY (1966). For deeper and more extensive studies of the Laplace–Beltrami operator see BERG (1969), HELGASON (1984), and FALLERT, GOODEY, and WEIL (in press).

1.3. Spherical Integration and Orthogonal Transformations

We first prove several auxiliary results regarding the integration of functions on S^{d-1}. It is interesting (although for our purpose not particularly important) that in most cases of spherical integration the pertinent statements remain true in the degenerate case when the integration is to be carried out over S^0. The integral is then to be interpreted as the sum of the function values of the two points of S^0.

Our first lemma relates integration over S^{d-1} to a particular integration over $[-1, 1]$. In this lemma, as well as in other formulas in this book, there appears a constant ϑ which is always defined by

$$\vartheta = \frac{d-3}{2}.$$

Lemma 1.3.1.

(i) *If N is a null set in $[-1, 1]$ with respect to Lebesgue measure on $\mathbf{E}^1$, then for every fixed $p \in S^{d-1}$ the set $U = \{u : u \in S^{d-1}, u \cdot p \in N\}$ is a null set with respect to Lebesgue measure σ on S^{d-1}.*

(ii) *If Φ is a bounded Lebesgue integrable function on $[-1, 1]$, and if p is a given point on S^{d-1}, then $\Phi(u \cdot p)$, considered as a function of u on S^{d-1}, is σ-integrable and*

$$\int_{S^{d-1}} \Phi(u \cdot p) d\sigma(u) = \sigma_{d-1} \int_{-1}^{1} \Phi(\zeta)(1-\zeta^2)^{\vartheta} d\zeta. \tag{1.3.1}$$

Proof. Let I_X denote the characteristic function of a set X; that means $I_X(x)$ equals 1 or 0 depending on whether $x \in X$ or $x \notin X$. If Φ is the characteristic function of a subinterval of $[-1, 1]$, then (1.3.1) is obtained by a calculus type computation of the area of a spherical zone on S^{d-1}. Summation and well-known properties of the Lebesgue integral then yield the validity of (1.3.1) for characteristic functions of unions of countably many disjoint intervals, in particular for step functions and characteristic functions of open sets.

To prove (i) let G_n be an open subset of $\mathbf{E}^1$ such that $N \subset G_n$ and the Lebesgue measure of G_n is less than $1/n$. Let U_n denote the open set $\{u : u \in S^{d-1}, u \cdot p \in G_n\}$. As already remarked, (1.3.1) is true for $\Phi = I_{G_n}$ and it follows that

$$\begin{aligned}\sigma(U_n) &= \int_{S^{d-1}} I_{G_n}(u \cdot p) d\sigma(u) = \sigma_{d-1} \int_{-1}^{1} I_{G_n}(t)(1-t^2)^{\vartheta} dt \\ &= \sigma_{d-1} \int_{G_n} (1-t^2)^{\vartheta} dt.\end{aligned}$$

Noting that $(1 - t^2)^{\vartheta}$ is an integrable function one finds that the last integral in this chain of inequalities will be less than any given $\epsilon > 0$ if n is large enough. Since $U \subset U_n$ (for all n), this implies that U is a null set.

To prove part (ii) we use the known fact that there are a uniformly bounded sequence of step functions g_i and a null set N in $\mathbf{E}^1$ such that

$$\lim_{i\to\infty} g_i(x) = \Phi(x) \qquad \text{if } x \notin N.$$

(In most textbooks, for example in RUDIN (1974), p. 57, this is proved for continuous functions rather than step functions; but it is also true for step functions since continuous functions on $[-1, 1]$ can be uniformly approximated by step functions.) If U is the set corresponding to N as stated in part (i) of the Lemma, it follows