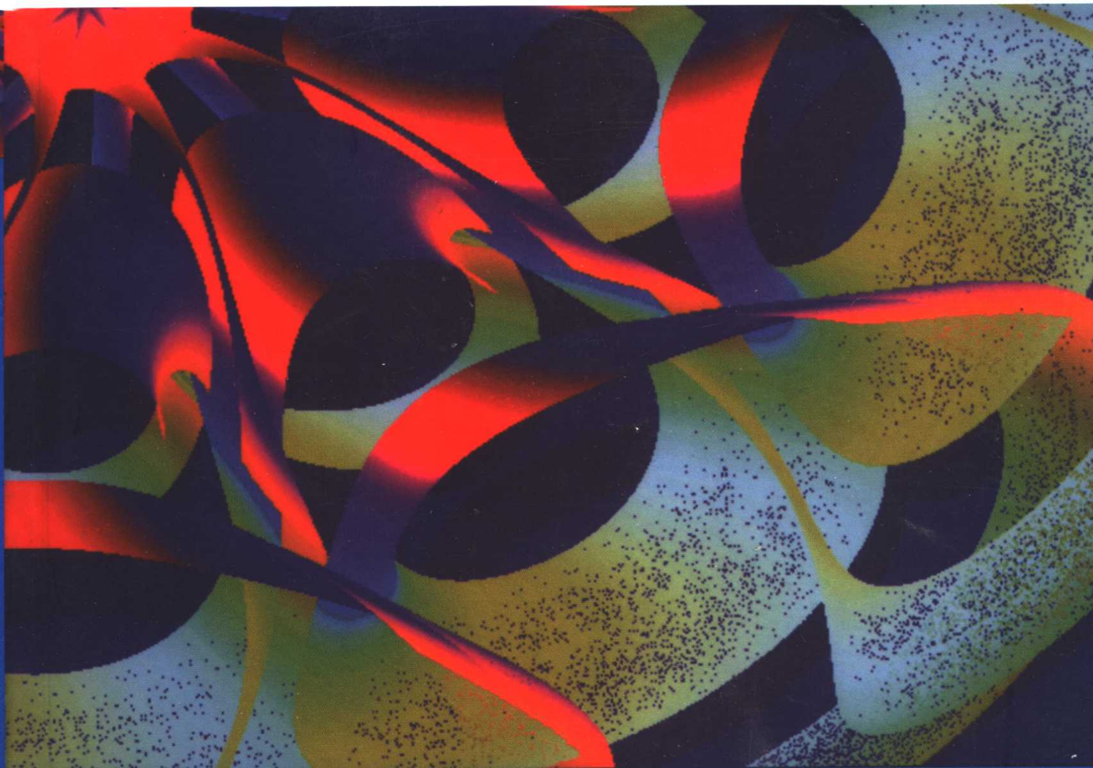

Nonlinear Waves, Solitons and Chaos

2nd Edition

Eryk Infeld & George Rowlands

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第 2 版

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Nonlinear Waves, Solitons and Chaos

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2nd edition

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Foreword to the first edition

The last few decades have seen three important developments in nonlinear classical physics, all of which extend across the board of physical disciplines. They have, however, received uneven coverage in the literature.

Perhaps the best known outburst of activity is associated with the soliton, and the most famous development here is the inverse scattering method which has been with us now for over twenty years. There are, however, several other, less known methods for treating solitons. Indeed these compact, single hump wave entities have been known to scientists for over a century and a half (it might be interesting to look through some old ships' log books!). Nevertheless, books on the subject tend to concentrate on the inverse scattering method.

The second much publicized development is a new understanding of some deterministic aspects of chaos as well as the various roads a physical system can take to reach a chaotic state. Established views are being revised and new concepts and indeed even universal constants are being found. These important new developments derive from a realization that complex chaotic behaviour can be described by simple equations. The field has now reached the stage where a summary of basic theory can be given, though applications to specific physical problems are largely at the research stage.

The third development is somewhat less well publicized. Over the last three decades or so, scientists working on fluid dynamics and plasma and solid state theory have developed a multitude of new methods to deal with nonlinear waves. Some of these people were aware of the shortcomings of our linear physics education even before the above-mentioned two developments brought them to the attention of the scientific community.

We believe that, although there are now a number of books on all three topics, the time has come to try to bring them together in one volume. Thus the present book documents the three important developments in classical physics jointly, and, when possible, points out the similarities of approach.

The authors' research interest over the past twenty years has been in fluid dynamics and plasma theory and this is reflected in the book. However, the main aim is to cover a wider range of nonlinear wave phenomena than hitherto. A few examples of what is done are: treatment of both surface and volume wave phenomena, including recent results (e.g. instabilities and their pictorial representation, wavelength doubling, wave dynamics in three dimensions, splitting of signals observed experimentally, the universal theory of wave envelope dynamics); new developments in soliton studies (e.g. many soliton experiments in rectangular, cylindrical and spherical devices and their theory); and a bringing together of theoretical and numerical results on various scenarios for reaching chaos. An example of what is not attempted is a coverage (or even mention) of the 100 or so instabilities found in plasmas and fluids. Instead we present the basic physics of a few of them, each representing a whole category in some sense. Thus, all in all, the ambition of the book could be summarized by the adage 'not many but much'. Some unsolved problems are indicated. References are extensive and exercises are given at the end of each chapter. Thus the more ambitious reader should be able to get into the field. On the other hand, little knowledge is assumed, thus also giving the general science graduate (or senior undergraduate), who would like to learn what these new developments are about, a chance to do so.

As one of us is based in Warsaw, an attempt has been made to do some justice to research performed in Poland and the Soviet Union.

The authors would like to thank Drs P. Frycz, P. Goldstein, T. Lenkowska, K. Murawski and L. Starkman for critical reading of the manuscript, and Professors P.N. Butcher*, J.P. Dougherty, A. Kuszell*, E. Kuznetsov, R. Raczka*, A.A. Skorupski, K.N. Stepanov and R. Zelazny for remarks on parts of the text. Additional thanks are due to Dr Frycz for preparation the material used in Sections 7.10 and A.1 and Figs. 9.2 and 9.3. We would also like to acknowledge a huge debt to Ms H. Gilder for typing the manuscript several times over. Finally, Cambridge University Press in the person of Dr S. Capelin has been both helpful and patient.

E. Infeld and G. Rowlands
Warwick

* Now deceased.

Foreword to the second edition

When this book was first published in 1990, it became more popular than we expected. A book club chose it as its Book of the Month. It was reprinted in 1992. Eight years have now gone by and we feel it is time for a proper revision. New results and references have been added. On the other hand, some chapters remain largely as they were, since we feel that the presentation of the basic ideas to be found there remains valid. Chapter 11 (Chapter 10 in the first edition) on chaotic phenomena is an example of this.

The only criticism anyone made to our faces was that we leaned too heavily on plasma physics and hydrodynamics for examples, whereas most phenomena and methods we consider have wider applications. These include optics, biology, solid state physics and other fields. This shortcoming has now been rectified to a certain extent. Also, a new chapter on soliton metamorphosis, including some colour plates, has been added (Chapter 10).

However, much of the text has been left as it was. Thus 'recent' should be read as recent in 1990. Some printing errors have been corrected. Once again, Dr Simon Capelin of the Cambridge University Press has been patient and helpful. Ms Lenkowska-Czerwińska spent a large portion of her time in Warwick helping us organize our material.

E. Infeld and G. Rowlands
Warwick

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1 Introduction

1.1 Occurrence of nonlinear waves and instabilities in Nature

This book is concerned with the propagation of waves and instabilities both linear and nonlinear, but concentrating on the latter.

The main advances in this subject have quite naturally come from studies involving fluids and more recently plasmas. The latter primarily because of the possibility of 'cheap, unlimited' (and hopefully safe) power from thermonuclear reactions. Everyone is of course familiar with waves on water if only being aware of the many instances where they provide examples of natural beauty. It is not so obvious that very similar waves can exist in a plasma which, to a good approximation, is usually a very dilute assembly of ions and electrons. We shall see later in this chapter that this is indeed so and fluids and plasmas have much in common. However, plasmas also show a much wider range of phenomena basically because they are composed of two or more components and also can be made strongly anisotropic by the introduction of magnetic fields, something that is not possible for simple fluids. This richer variety of phenomena has also been a reason why plasmas have had more than their share of attention.

The above remarks notwithstanding, there are numerous media other than plasmas and fluids which can support waves and/or propagate instabilities. As we will see, some of these are more intriguing than others.

1.1.1 Nonlinear phenomena in our everyday experience

As most of us are aware, waves generated by the wind can propagate across a field of corn. In this case the microscopic model is that of the ears moving, due to the stalk bending, in an harmonic manner, and ear interacting with adjacent ear only when in contact (a hard sphere potential). On a macroscopic level the corn can be considered as a dense fluid and now with the moving air flow over it one has the classic situation of a Kelvin–Helmholtz instability (Chapter 4).

The wind drives the instability and the stalks of corn bend in an analogous manner to how water waves are formed on lakes by the wind. The nonlinear requirement is different, in that if the bending of the stalk is too great it will break and produce a permanent record of the wave. Kelvin–Helmholtz type instabilities occur in plasma physics and they have been controlled to a

Figure 1.1 Two examples of herring-bone cloud formation. (After R. Scorer, *Clouds of the world*, Lothian, Melbourne (1972).)

certain extent by introducing perpendicular periodicities. The intriguing question naturally arises if similar conditions would stabilize the motion of corn heads and stop the breaking of the stalks. By analogy this could be done by planting trees periodically spaced in a line (or lines) perpendicular to the usual direction of the wind. Another example where Nature leaves a permanent record of a surface wave instability (at least until diffusion processes slowly remove it) is the so-called herring-bone cloud formation. Here the sky is broken up into alternate bands of cloud (high density of moisture regions) and apparently clear regions (low density regions). The moisture in this case is responsible for the permanent record, two layers of air moving relative to each other giving rise to the Kelvin–Helmholtz instability (Chapter 4) and subsequent nonlinear effects (Fig. 1.1).

Television coverage of soccer matches usually shows, incidentally, the swaying of the crowd. This is seen as a wave moving through the stadium. Unfortunately, in some circumstances this wave can build up and those near the barriers can get crushed.

Lighthill and Whitham (1955) and Richards (1956) have considered the flow of cars (discrete objects) in a fluid context and explained phenomena such as the effects of traffic lights in terms of the propagation of waves and, in particular, shocks.

The conclusion to be reached from these quite disparate examples is that they can be effectively studied in terms of the propagation of waves and instabilities, leading to nonlinear effects, in continuous media. If the natural wavelength is large compared to the underlying microscopic length, the above picture should be applicable.

1.1.2 *Nonlinear phenomena in the laboratory*

Waves in solids have received considerable attention both at the microscopic (atomic) and the macroscopic (continuous) level. The most interesting phenomenon concerning the propagation of disturbances in solids at a microscopic level is the effect of anisotropies and non-homogeneities in the media. Until recently, nonlinear effects in solids have received little attention, as the energy needed to produce them is very large. However, with the advent of intense power sources such as lasers it has been possible to show that the flow of heat in solids is closely related to the flow of solitons. The basic relationships between soliton amplitude, width and velocity have been verified in this context (Section 1.2 and, in some detail, Chapters 5 and 7).

At the microscopic level it is usually necessary to quantize the system and instead of talking about sound waves one talks about phonons. However, because of the lattice periodicity the phonons have a dispersive nature. Most interesting phenomena, such as thermal conductivity, depend on phonon–phonon interaction. Until recently, such interaction has been studied in what could be called weakly nonlinear theories. However, a major breakthrough was made when it was realized that a number of phenomena could best be explained by introducing solitons as elementary excitations. Then it was found that a statistical mechanics based on weakly interacting solitons and phonons gave better results than previous theories which attempted to consider phonon–phonon interaction outside the control of a weak interaction.

Many years ago, Fermi, Ulam and Pasta (1965), though the actual work was performed much earlier and published in a Los Alamos report in 1955, studied numerically the problem of the strong interaction of phonons. They found, somewhat perplexingly at that time, that the

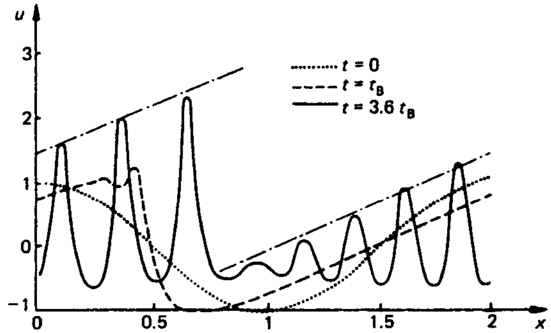

Figure 1.2 Evolution of an initially periodic profile, cosπx, as given by the Korteweg–de Vries equation (1.2.8). The breaking time for the wave profile (when the third term is neglected) is t_B. From Zabusky and Kruskal (1965). After a while, patterns roughly repeat themselves.

phonons did not come to thermal equilibrium, but rather they underwent nearly periodic variations. See Weissert (1998).

Much later Zabusky and Kruskal (1965) showed that this was the correct behaviour and *could* be explained in terms of solitons in a space periodic medium, Fig. 1.2. (A name given to reflect their quantal nature.) Nowadays people realize that solitons are not *necessary* to explain this phenomenon (Thyagaraja (1983)) Chapter 5. A theory that relies on the interaction of a small number of periodic modes also exists, see Infeld (1981c). Solitons are special wave pulses which interact with one another so as to keep their basic identity and so that they act as particles. Now the soliton has come of age in its mother subject, though in the meantime promising to be a useful concept in many other branches of physics, most of them being well out of the range of quantum effects. One of the first detailed experimental verifications of soliton type behaviour was in the study of nonlinear ion acoustic waves.

Davydov (1978), (1985) has applied some of the rules of solid state physics to the transport of energy down protein chains. He assumed that the idea of soliton propagation is relevant to a study of chemical changes taking place in long protein molecules. This leads to the transfer of ATP (adenosine triphosphate) and could be the basis for an understanding of muscle contraction. However, Davydov's theory is one-dimensional, whereas proteins are three-dimensional. Also, his mechanism for energy transport has been criticized. Zorski and Infeld (1997) approach protein dynamics by using a quadrilateral chip model. The dynamics were described by continuum equations in three dimensions. They obtained a helical structure and demonstrated stability. Solitons do appear in special cases, but have limited importance in their theory. Thus, the role of soliton dynamics in understanding protein chains still seems to be an open question.

The motion of electrons in solids is surprisingly well understood in terms of a simple Drude picture where the current J is linearly related to the electric field E by $J = (ne^2\tau/m)E$. Here n is the electron density, e its charge, and τ the mean free time. The effective mass 'm' takes into account the presence of the periodic nature of atomic structure. Normally all the quantities are constants and we simply have Ohm's law. However, it can happen, particularly in semiconductors, that if E is large enough the electron can be excited to a higher band which can have a different value of m. Thus the conductivity $\sigma(= ne^2\tau/m)$ is dependent on E and Ohm's law is no longer linear. If the mass is larger in the higher band, then an increase in E causes a decrease in J and thus we have a negative differential resistance. Obviously this is an instability mechanism and such a mechanism is observed in GaAs (gallium arsenide). The instability itself leads to the propagation of nonlinear stable pulses called Gunn domains, analogues to a soliton. These have been observed and are in fact the bases of many modern day oscillators. See Butcher (1967) for a review of Gunn domains; and Butcher and Rowlands (1968) for a study of the stability of the domain.

Another instability, leading to the propagation of nonlinear pulses that are also kinsmen of solitons and have been studied in the general area of solid state physics, is that associated with the acousto-electric effect. Here the instability mechanism arises because of a piezo-electric coupling between the propagation of sound waves and the flow of electrons. A nonlinear pulse can propagate down a crystal, for example of CdS or ZnO, but reach such an amplitude as to cause permanent distortion to the crystal. For a brief account of the general theory and for a discussion of the instability and nonlinear effects see Pawlik and Rowlands (1975).

It took a long time for it to be accepted that a homogeneous mixture of chemicals could lead to a periodic time variation in the concentration of a particular chemical or to an inhomogeneous spatial separation of the chemicals. Turing demonstrated that nonlinear chemical reactions together with diffusion could lead to a spatial separation of the chemicals and explored the idea in connection with the theory of morphogenesis, the formation of life. Zhabotinsky (1964) and later Zhabotinsky and Belousov found experimentally that a homogeneous mixture of certain chemicals could lead to a time periodic variation of the colour of the mixture. Later experiments showed that the same mechanism could lead to spatial colour patterns. All these phenomena can be explained in terms of nonlinear waves in time and space that have the inherent stability of a soliton.

In this revised version of the book we will look at nonlinear phenomena in new contexts, such as laser theory and biology. Waves following from discrete equations are also a novelty, as are illustrations of soliton metamorphosis (solitons are a kind of compact wave, about which more in Subsection 1.2).

1.2 Universal wave equations

It has now been realized that the study of many different types of waves in numerous media can often be based on a few universal nonlinear equations. These replace the usual linear wave equations, such as

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = 0.$$

1.2.1 The Korteweg–de Vries and Kadomtsev–Petviashvili equations and a first look at solitons

Most classical media propagate longitudinal plane waves at or near a characteristic velocity c (acoustic-type waves). These waves are, for very small amplitude, given by

$$u = a\cos(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{1.2.1}$$

where ω is taken to be a function of $\mathbf{k}$, the form of which is dictated by the medium. When this medium is isotropic, we expect ω to depend on the modulus k only. All acoustic waves are such that, for small k,

$$\omega^2 = c^2k^2 + \cdots, \tag{1.2.2}$$

where c is the velocity of sound or another velocity specific to the medium. Thus long-wave acoustic modes propagate with little or no dispersion and the signal (group) velocity $\partial\omega/\partial\mathbf{k}$ is almost equal to the phase velocity c:

$$\frac{\partial\omega}{\partial\mathbf{k}} \simeq \omega k^{-2}\mathbf{k} = ck^{-1}\mathbf{k}. \tag{1.2.3}$$

However, dispersion will come in for k other than very small and it is perhaps more natural for the signal to lag behind the phase. This stipulation and symmetry suggest that the simplest possible correction to (1.2.2) is a negative quartic term in k:

$$\omega^2 = c^2k^2 - \beta^2k^4. \tag{1.2.4}$$

We have arrived at a very general small $\mathbf{k}$ dispersion relation. It covers all isotropic media that propagate acoustic modes such that the signal lags behind the phase. For the moment we will consider one space dimension, returning to general $\mathbf{k}$ later on. If we follow a wave propagating from left to right (admittedly thus losing some generality) we have

$$\omega = ck - (\beta^2/2c)k^3 + \cdots. \tag{1.2.5}$$

We can look at the wave behaviour in a coordinate system moving with velocity c and renormalize lengths so as to get rid of the $\beta^2/2c$ coefficient. Thus, in the new system,

$$\omega^* = -k^3, \tag{1.2.6}$$

corresponding to the following differential equation for u:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1.2.7}$$

This equation has several drawbacks. For one, it is not Galilean-invariant (this is partly due to our choice of coordinate system). It also leads to spreading of all finite extent initial profiles $u(x, 0)$ (dispersion). However, if we replace the first term by the more general convective derivative, we remove both these shortcomings. Thus we suggest

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.2.8}$$

as a more adequate equation. This is known as the Korteweg–de Vries (KdV) equation (1895) and will be derived rigorously in two physical contexts in Chapter 5 (water surface gravity waves and ion acoustic waves in a plasma). Some subsequent developments are reviewed by Miura (1976) and Miles (1981). However, it is already seen here to be the simplest possible *unidirectional* wave equation including dispersion and nonlinearity, but not dissipation. This allows us to hope for a stationary, pulse-like solution to exist if the nonlinearity, leading to wave steepening, can just