Søren R. K. Nielsen
Zili Zhang

Stochastic Dynamics

Aarhus University Press

Buildings, bridges and tunnels are continuously exposed to forces of nature such as wind, waves, earthquakes and even traffic. Because these forces are hard to predict and model, it is also hard to predict how these structures respond.

Stochastic Dynamics introduces a way of modeling forces and responses using stochastic systems, providing a method of statistical analysis where precise predictions are impossible. Including topics such as stochastic processes, stochastic vibration theory of linear structures, reliability theory of dynamic structures and Monte Carlo techniques, this book explains stochastic structural dynamics and how to apply it to modern structures.

Søren R.K. Nielsen is a Professor in the Department of Civil Engineering, Aalborg University, Denmark. His research areas include stochastic and nonlinear dynamics, reliability theory and vibration control.

Zili Zhang is an Assistant Professor in the Department of Engineering, Aarhus University, Denmark. His research areas include structural dynamics, vibration control and wind energy.

9 788771 842326

Aarhus University Press

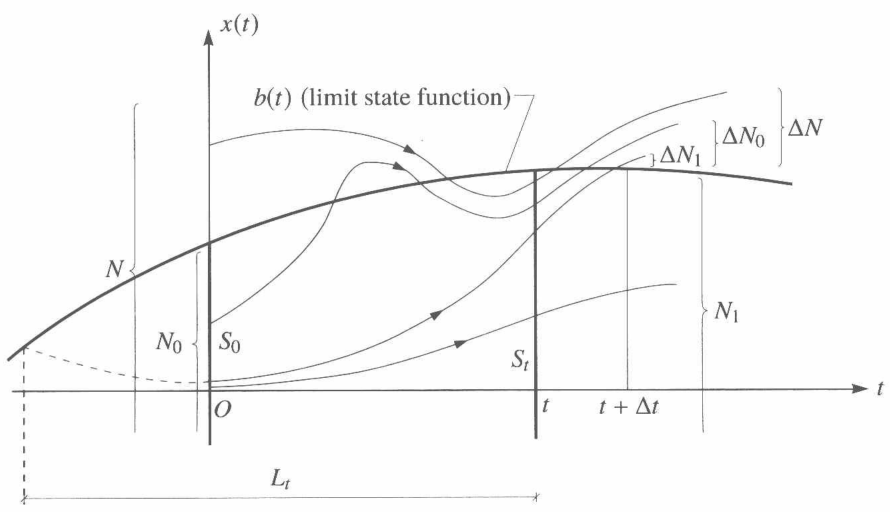
Søren R. K. Nielsen and Zili Zhang

Stochastic Dynamics

STOCHASTIC DYNAMICS

STOCHASTIC DYNAMICS

Søren R. K. Nielsen
Zili Zhang



Aarhus University Press

Stochastic Dynamics

Cover: Nethe Ellinge Nielsen, Trefold
Publishing editor: Simon Olling Rebsdorf

Printed on Amber Graphic by Narayana Press

Printed in Denmark 2017
ISBN 978 87 7184 232 6

AARHUS UNIVERSITY PRESS
Finlandsgade 29
DK-8200 Aarhus N
Denmark
www.unipress.dk

Weblinks were active when the book was printed. They may no longer be active.

INTERNATIONAL DISTRIBUTION

UK & Eire
Gazelle Book Services Ltd.
White Cross Mills, Hightown
Lancaster LA1 4XS, England
www.gazellebooks.com

North America
ISD
70 Enterprise Drive
Bristol, CT 06010
USA
www.isddistribution.com

FSC
www.fsc.org
MIX
Paper from responsible sources
FSC® C010651

Contents

PREFACE

The present textbook has been written based on previous lecture notes for a course on stochastic vibration theory given in the autumn semester at Aarhus University for M.Sc. students in structural engineering.

In chapter 1, the basic assumptions of the random vibration theory are emphasized. In chapters 2 and 3, pertinent results of stochastic variables and stochastic processes have been indicated. Chapter 4 deals with the stochastic response analysis of single-degree-of-freedom, multi-degree-of-freedom and continuous linear structural systems. In principle, an introductory course on linear structural dynamics is presupposed. However, in order to make this textbook self-contained, short reviews of the most important results of linear deterministic vibration theory have been included in the start of the relevant sub-sections. Chapter 5 outlines the reliability theory for dynamically excited civil engineering structures, i.e., reliability theory for narrow-banded response processes. Finally, Chapter 6 gives an introduction to Monte Carlo simulation techniques, which become increasingly important and useful as the computers become more and more powerful.

Aarhus University, December 2016

Søren R. K. Nielsen
Zili Zhang

CHAPTER 1
INTRODUCTION

A structural system may be considered as an ensemble of mass particles, each of which is exposed to external loading, and to internal forces from neighbouring mass particles. If the initial conditions, and the external and internal forces are perfectly described for all mass particles, the motion of the system can in principle be determined from *Newton's 2nd law of motion*, leaving no room for any indeterminism. This is the principle of *causality* or *determinism*, postulated in classical physics.

In practice, neither the initial conditions, nor the external and internal forces can be perfectly determined for the mass particles. Usually, the design loads of the structure are some future extreme loadings, which cannot be specified in space and time in the sense presumed above. Even if this was the case, the internal forces cannot be observed or specified. Instead, these internal forces have to be theoretically determined by a mathematical model. In the continuum mechanics approach, the internal forces are expressed in terms of stresses. Significant modelling errors may stem from the constitutive equation which relates the stress components to the motions of the mass particles, and also from certain kinematical approximations rendering analytical solutions possible (e.g., classical beam-, plate-, and shell theories). These errors or approximations introduce uncertainty into the predicted motions of the mass particles. The accuracy of the prediction depends on the level of uncertainty with which the *initial values*, and the external and internal forces can be specified.

Generally, uncertainties are classified as either aleatoric or epistemic.

Aleatoric uncertainties are uncertainties that cannot be removed or reduced by any means. Typical examples are external dynamic natural loads from winds, waves and earthquakes, but also some man made loads such as traffic loads.

Epistemic uncertainties can be removed or reduced to a certain extent. An example is the measurement uncertainties in active feedback vibration control problems, which can be reduced by better measurement equipments or filtering procedures. Another example is the structural modelling uncertainty, which can be reduced by better numerical or analytical models. Epistemic uncertainties can only be reduced to a certain extent, after which they should be considered as aleatoric.

Both aleatoric and epistemic uncertainties are quantified by stochastic models.

The subject of *stochastic mechanics* is the quantification of uncertainty of the structural response based on the quantified uncertainty of the initial values and the external and internal forces. The word "stochastic" (from greek *stochas'mos* = presumerable), with the usual meaning that something is happening by chance, is in a way misleading, because nothing in nature is considered to happen by chance. Stochastic mechanics should merely be interpreted as a tool for quantification of uncertainty of some quantities in the mechanical problem. If these quantities are observed or controlled, the uncertainty is removed or reduced. This would not be possible if an inherent indeterminism is present in nature.

As stated, a mathematical model must be adapted for the determination of internal forces. The quantification of the errors associated with this choice is one of the unsolved problems in stochastic mechanics. The frequently mentioned suggestion to evaluate the performance of the model against a more elaborated model provides insight into the accuracy of the selected model, but does not carry the uncertainty of the chosen model into an operable scheme, where the modelling uncertainty can be weighted along with the uncertainties associated with the initial values and the external loadings.

Even for a well-defined mathematical model for the internal forces, additional uncertainty can be introduced if the parameters of the model cannot be properly calibrated. As an example, the beam in Fig. 1-1 is assumed to be modelled using *Bernoulli-Euler beam theory*. However, the bending stiffness $EI(x)$ in this model may still be uncertain. This is the case for reinforced concrete members under dynamic loading, where cracked and uncracked sections may alternate in an uncontrolled way along the beam. In contrary to the more fundamental modelling problem for the internal forces, parameter uncertainty can easily be treated within the realm of stochastic mechanics.

In stochastic mechanics, the uncertainties of the external loadings and possible parameter fields of the mathematical model for the internal forces are usually modelled as *stochastic variables* or *stochastic processes*. The stochastic structure of these stochastic variables and stochastic processes is estimated from available measurements, statistical inference and engineering

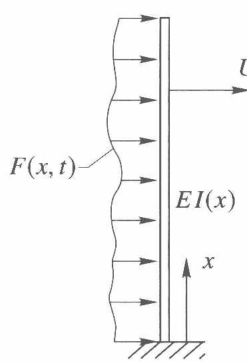


Fig. 1–1 Bernoulli-Euler beam with uncertain bending stiffness.

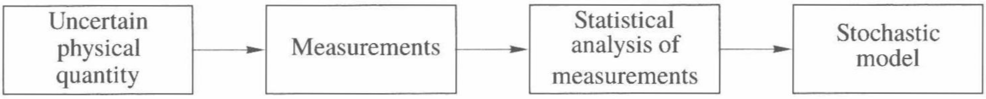


Fig. 1–2 Stochastic modelling of uncertain physical quantity.

judgement. However, this statistical calibration process is not considered a subject of the stochastic response analysis, which assumes that such a calibration has already been performed. The various steps in a stochastic modelling problem have been illustrated in Fig. 1-2. Then, the stochastic analysis merely concentrates on the determination of the stochastic structure of the response processes in terms of the corresponding stochastic structure of the external loading processes and of possible parameter fields associated with the adapted mathematical model.

Dynamic loadings on civil engineering structures are all characterized as highly uncertain. For most structures, the uncertainty of these loadings is much larger than the uncertainty associated with conventional mathematical models for structural analysis, even though this is not always true. For example, the modelling of the dissipation mechanism of mechanical energy into heat may be related with large uncertainty. Nevertheless, in a large number of cases, the modelling uncertainty can be disregarded on condition that a conventional and well-tested mathematical model for structural analysis is adopted. This is the basic assumption in classical stochastic dynamic theory, which is also the basis of the present book. The adopted structural model may be linear or non-linear. However, in the following, only linear structural models are considered for which the *superposition theorem* is applicable. Then, the structural response is determined as a linear functional of the external loading in terms of a memory integral known as *Duhamel's integral*. Since the structure is considered deterministic, no uncertainty is related to the kernel of the memory integral (the so-called *impulse response function*).

The ultimate aim of any engineering analysis is to make decisions on the ability of a certain proposed structural design to fulfil its purpose. Within the context of stochastic mechanics, the decision parameter of interest is given by the *probability of failure*. If it is too high, the draft proposal must be changed. In the present book, various approximate methods have been indicated for the determination of this quantity for dynamically excited structures. In any case, the reliability analysis assumes a preceding stochastic dynamic analysis which provides the necessary parameters needed.

CHAPTER 2
STOCHASTIC VARIABLES

2.1 Introduction to stochastic variables

A *stochastic variable* X is a real function defined on a *sample space* Ω, with the property that any subset of Ω of the following type

$$A_x = \{\omega \in \Omega \mid X(\omega) \le x\} \tag{2–1}$$

is an *event*. This means that a probability $P(A_x)$ can be assigned to the event A_x. $P(\cdot)$ is the *probability function* mapping all events $A \subseteq \Omega$ into the interval [0, 1].

Next, the *probability distribution function* $F_X(x)$ of the stochastic variable X is defined as:

$$F_X(x) = P(A_x) \tag{2–2}$$

The following abbreviated notation will be used in the text: $F_X(x) = P(X \le x)$.

It is noticed that $A_{-\infty} = \emptyset$ and $A_{+\infty} = \Omega$, and $P(\emptyset) = 0$, $P(\Omega) = 1$, where $\emptyset$ signifies an *impossible event*. Hence the following results hold:

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1 \tag{2–3}$$

Since $x_1 \le x_2 \Rightarrow A_{x_1} \subseteq A_{x_2}$, one has $x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$. This establishes the probability distribution function as a non-decreasing function of x.

In what follows, it is assumed that $F_X(x)$ is continuous and piecewise differentiable. The derivative of the probability distribution function, called the *probability density function*, is defined as:

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{2–4}$$

Then, the probability of the event $\{x_1 < X(\omega) \le x_2\}$ becomes:

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\, dx \tag{2–5}$$

Table 2-1 Probability density functions for some continuous stochastic variables.

Distribution	Probability density function	Short notation
Uniform	$f_X(x) = \begin{cases} 0, & x < a \\ \frac{1}{b-a}, & a \le x \le b \\ 0, & x > b \end{cases}$	$X \sim U(a, b)$
Normal	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$	$X \sim N(\mu, \sigma^2)$
Gamma	$f_X(x) = \begin{cases} 0, & x < 0 \\ \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\beta x}, & x \ge 0 \end{cases}$	$X \sim G(\alpha, \beta)$
Exponential	$f_X(x) = \begin{cases} 0, & x < 0 \\ \beta e^{-\beta x}, & x \ge 0 \end{cases}$	$X \sim E(\beta)$
Rayleigh	$f_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), & x \ge 0 \end{cases}$	$X \sim R(\sigma^2)$
Weibull	$f_X(x) = \begin{cases} 0, & x < x_0 \\ \frac{\alpha}{x_1}(\frac{x-x_0}{x_1})^{\alpha-1} \exp(-(\frac{x-x_0}{x_1})^\alpha), & x \ge x_0 \end{cases}$	$X \sim W(x_0, x_1, \alpha)$

The probability density function is nonnegative, and due to Eq. (2-3) the following normalization condition prevails:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{2–6}$$

A stochastic variable is completely specified by its probability distribution function or its probability density function. In Table 2-1, some well-known probability density functions are listed.

Let X and Y be stochastic variables defined on the same sample space Ω. The column vector $[X, Y]^T$ is termed a *two-dimensional stochastic vector*. Considering the events $A_x = \{X \le x\}$ and $B_y = \{Y \le y\}$, the intersection $A_x \cap B_y$ is also an event, and consequently the probability $P(A_x \cap B_y)$ can be defined as well. Obviously, this is a function of the pair (x, y), i.e., $F_{XY}(x, y)$, which is termed the *joint probability distribution function* of the two-dimensional stochastic vector $[X, Y]^T$. The following short notation is applied:

$$F_{XY}(x, y) = P(X \le x \wedge Y \le y) \tag{2–7}$$