SHAFAREVICH

BASIC ALGEBRAIC GEOMETRY

Schemes and Complex Manifolds

Second Edition

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Basic Algebraic Geometry 2

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Preface to Books 2 and 3

Books 2 and 3 correspond to Chap. V–IX of the first edition. They study schemes and complex manifolds, two notions that generalise in different directions the varieties in projective space studied in Book 1. Introducing them leads also to new results in the theory of projective varieties. For example, it is within the framework of the theory of schemes and abstract varieties that we find the natural proof of the adjunction formula for the genus of a curve, which we have already stated and applied in Chap. IV, 2.3. The theory of complex analytic manifolds leads to the study of the topology of projective varieties over the field of complex numbers. For some questions it is only here that the natural and historical logic of the subject can be reasserted; for example, differential forms were constructed in order to be integrated, a process which only makes sense for varieties over the (real or) complex fields.

Changes from the First Edition

As in the Book 1, there are a number of additions to the text, of which the following two are the most important. The first of these is a discussion of the notion of the algebraic variety classifying algebraic or geometric objects of some type. As an example we work out the theory of the Hilbert polynomial and the Hilbert scheme. I am very grateful to V. I. Danilov for a series of recommendations on this subject. In particular the proof of Chap. VI, 4.3, Theorem 3 is due to him. The second addition is the definition and basic properties of a Kähler metric, and a description (without proof) of Hodge's theorem.

Prerequisites

Varieties in projective space will provide us with the main supply of examples, and the theoretical apparatus of Book 1 will be used, but by no means all of it. Different sections use different parts, and there is no point in giving exact indications. References to the Appendix are to the Algebraic Appendix at the end of Book 1.

Prerequisites for the reader of Books 2 and 3 are as follows: for Book 2, the same as for Book 1; for Book 3, the definition of differentiable manifold, the basic theory of analytic functions of a complex variable, and a knowledge of homology, cohomology and differential forms (knowledge of the proofs is not essential); for Chap. IX, familiarity with the notion of fundamental group and the universal cover. References for these topics are given in the text.

Suggestions for Further Reading

Some references for further reading are included in the text. The reader who would like to continue the study of algebraic geometry is recommended the following books.

For scheme theory, the cohomology of algebraic coherent sheaves and its applications, see: Hartshorne [35], especially Chap. III.

For the Riemann–Roch theorem. An elementary proof for curves is given in the book: W. Fulton, Algebraic curves, Springer.

For the general case, see any of the following:

A. Borel and J.-P. Serre, Le théorème de Riemann-Roch, Bull. Soc. Math. France, **86** (1958), 97–136. Or

Yu. I. Manin, Lectures on the K-functor in algebraic geometry, Uspekhi Mat. Nauk **24**:5 (1969), 3–86. English translation in: Russian Math. Surveys, **24**:5 (1969), 1–89. Or

W. Fulton and S. Lang, Riemann-Roch algebra, Springer, 1985.

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BOOK 2

Schemes and Varieties

Chapter V. Schemes

In this chapter, we return to the starting point of all our study – the notion of algebraic variety – and attempt to look at it from a more general and invariant point of view. On the one hand, this leads to new ideas and methods that turn out to be exceptionally fertile even for the study of the quasiprojective varieties we have worked with up to now. On the other, we arrive in this way at a generalisation of this notion that vastly extends the range of application of algebraic geometry.

What prompts the desire to reconsider the definition of algebraic variety from scratch? Recalling how affine, projective and quasiprojective varieties were defined, we see that in the final analysis, they are all defined by systems of equations. One and the same variety can of course be given by different equations, and it is precisely the wish to get away from the fortuitous choice of the defining equations and the embedding into an ambient space that leads to the notion of isomorphism of varieties. Put like this, the framework of basic notions of algebraic geometry is reminiscent of the theory of finite field extensions at the time when everything was stated in terms of polynomials: the basic object was an equation and the idea of independence of the fortuitous choice of the equation was discussed in terms of the "Tschirnhaus transformation". In field theory, the invariant treatment of the basic notion considers a finite field extension $k \subset K$, which, although it can be represented in the form $K = k(\theta)$ with $f(\theta) = 0$ (for a separable extension), reflects properties of the equation $f = 0$ invariant under the Tschirnhaus transformation. As another parallel, one can point to the notion of manifold in topology, which was still defined right up to the work of Poincaré as a subset of Euclidean space, before its invariant definition as a particular case of the general notion of topological space.

The nub of this chapter and the next will be the formulation and study of the "abstract" notion of algebraic variety, independent of a concrete realisation. This idea thus plays the role in algebraic geometry of finite extensions in field theory or of the notion of topological space in topology.

The route by which we arrive at such a definition is based on two observations concerning the definition of quasiprojective varieties. In the first place, the basic notions (for example, regular map) are defined for quasiprojective varieties starting from their covers by affine open sets. Secondly, all the prop-

erties of an affine variety X *are reflected in the ring* $k[X]$, which is associated with it in an invariant way. These arguments suggest that the general notion of algebraic variety should in some sense reduce to that of affine variety; and that in defining affine varieties, one should start from rings of some special type, and define the variety as a geometric object associated with the ring.

It is not hard to carry out this program: in Chap. I we studied in detail how properties of an affine variety X are reflected in its coordinate ring $k[X]$, and this allows us to construct a definition of the variety X starting from some ring, which turns out after the event to be $k[X]$. However, proceeding in this way, we can get much more than the invariant definition of an affine algebraic variety. The point is that the coordinate ring of an affine variety is a very special ring: it is an algebra over a field, is finitely generated over it, and has no nilpotent elements. However, as soon as we have worked out a definition of affine variety based on some ring A satisfying these three conditions, the idea arises of replacing A in this definition by a completely arbitrary commutative ring. We thus arrive at a far-reaching generalisation of affine varieties. Since the general definition of algebraic variety reduces to that of an affine variety, it also is the subject of the same degree of generalisation. The general notion which we arrive at in this way is called a scheme.

The notion of scheme embraces a circle of objects incomparably wider than just algebraic varieties. One can point to two reasons why this generalisation has turned out to be exceptionally useful both for "classical" algebraic geometry and for other domains. First of all, the rings appearing in the definition of affine scheme are not now restricted to algebras over a field. For example, this ring may be a ring such as the ring of integers $\mathbf{Z}$, the ring of integers in an algebraic number field, or the polynomial ring $\mathbf{Z}[T]$. Introducing these objects allows us to apply the theory of schemes to number theory, and provides the best currently known paths for using geometric intuition in questions of number theory. Secondly, the rings appearing in the definition of affine scheme may now contain nilpotent elements. Using these schemes allows us, for example, to apply in algebraic geometry the notions of differential geometry related with infinitesimal movements of points or subvarieties $Y \subset X$, even when X and Y are quasiprojective varieties. And we should not forget that, as a particular case of schemes, we get the invariant definition of algebraic varieties which, as we will see, is much more convenient in applications, even when it does not lead to any more general notion.

Since we expect that the reader already has sufficient mastery of the technical material, we drop the usual "from the particular to the general" style of our book. Chap. V introduces the general notion of scheme and proves its simplest properties. In Chap. VI we define "abstract algebraic varieties", which we simply call varieties. After this, we give a number of examples to show how the notions and ideas introduced in this chapter allow us to solve a number of concrete questions that have already occurred repeatedly in the theory of quasiprojective varieties.

1. The Spec of a Ring

1.1. Definition of Spec A

We start out on the program sketched in the introduction. We consider a ring A, always assumed to be commutative with 1, but otherwise arbitrary. We attempt to associate with A a geometric object, which, in the case that A is the coordinate ring of an affine variety X, should take us back to X. This object will at first only be defined as a set, but we will subsequently give it a number of other structures, for example a topology, which should justify its claim to be geometric.

The very first definition requires some preliminary explanations. Consider varieties defined over an algebraically closed field. If we want to recover an affine variety X starting from its coordinate ring $k[X]$, it would be most natural to use the relation between subvarieties $Y \subset X$ and their ideals $\mathfrak{a}_Y \subset k[X]$. In particular a point $x \in X$ corresponds to a maximal ideal $\mathfrak{m}_x$, and it is easy to check that $x \mapsto \mathfrak{m}_x \subset k[X]$ establishes a one-to-one correspondence between points $x \in X$ and the maximal ideals of $k[X]$. Hence it would seem natural that the geometric object associated with any ring A should be its set of maximal ideals. This set is called the *maximal spectrum* of A and denoted by m-Spec A. However, in the degree of generality in which we are now considering the problem, the map $A \mapsto$ m-Spec A has certain disadvantages, one of which we now discuss.

It is obviously natural to expect that the map sending A to its geometric set should have the main properties that relate the coordinate ring of an affine algebraic variety with the variety itself. Of these properties, the most important is that homomorphisms of rings correspond to regular maps of varieties. Is there a natural way of associating with a ring homomorphism $f\colon A \to B$ a map of m-Spec B to m-Spec A? How in general does one send an ideal $\mathfrak{b} \subset B$ to some ideal $\mathfrak{b} \subset A$? There is obviously only one reasonable answer, to take the inverse image $f^{-1}(\mathfrak{b})$. But the trouble is that the inverse image of a maximal ideal is not always maximal. For example, if A is a ring with no zerodivisors that is not a field, and $f\colon A \hookrightarrow K$ an inclusion of A into a field, then the zero ideal (0) in K is the maximal ideal of K, but its inverse image is the zero ideal (0) in A, which is not maximal.

This trouble does not occur if instead of maximal ideals we consider prime ideals: it is elementary to check that the inverse image of a prime ideal under any ring homomorphism is again prime. In the case that $A = k[X]$ is the coordinate ring of an affine variety X, the set of prime ideals of A has a clear geometric meaning: it is the set of irreducible closed subvarieties of X (points, irreducible curves, irreducible surfaces, and so on). Finally, for a very large class of rings the set of prime ideals is determined by the set of maximal ideals (see Ex. 8). All of this motivates the following definition.

Definition. The set of prime ideals of A is called its *prime spectrum* or simply *spectrum*, and denoted by $\operatorname{Spec} A$. Prime ideals are called *points* of $\operatorname{Spec} A$.

Since we only consider rings with a 1, the ring itself is not counted as a prime ideal. This is in order that the quotient ring A/P by a prime ideal should always be an integral domain, that is, a subring of a field (with $0 \neq 1$). Every nonzero ring A has at least one maximal ideal. This follows from Zorn's lemma (see for example Atiyah and Macdonald [7], Theorem 1.3); thus $\operatorname{Spec} A$ is always nonempty for $A \neq 0$.

We have already discussed the geometric meaning of $\operatorname{Spec} A$ when $A = k[X]$ is the coordinate ring of an affine variety. We consider some other examples.

Example 1. $\operatorname{Spec} \mathbf{Z}$ consists of the prime ideals (2), (3), (5), (7), (11), ..., and the zero ideal (0).

Example 2. Let $\mathcal{O}_x$ be the local ring of a point x of an irreducible algebraic curve. Then $\operatorname{Spec} \mathcal{O}_x$ consists of two points, the maximal ideal and the zero ideal.

Consider a ring homomorphism $\varphi\colon A \to B$. In what follows we always consider only homomorphisms that take $1 \in A$ into $1 \in B$. As we remarked above, the inverse image of any prime ideal of B is a prime ideal of A. Sending a prime ideal of B into its inverse image thus defines a map

$$^a\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A,$$

called the *associated map* of φ.

As a useful exercise, the reader might like to think through the map $\operatorname{Spec}(\mathbf{C}[T]) \to \operatorname{Spec}(\mathbf{R}[T])$ associated with the inclusion $\mathbf{R}[T] \hookrightarrow \mathbf{C}[T]$.

Example 3. We consider the ring $\mathbf{Z}[i]$ with $i^2 = -1$, and try to imagine its prime spectrum $\operatorname{Spec}(\mathbf{Z}[i])$, using the inclusion map $\varphi\colon \mathbf{Z} \to \mathbf{Z}[i]$. This defines a map

$$^a\varphi\colon \operatorname{Spec}(\mathbf{Z}[i]) \to \operatorname{Spec} \mathbf{Z}.$$

We write $\omega = (0) \in \operatorname{Spec} \mathbf{Z}$ and $\omega' = (0) \in \operatorname{Spec}(\mathbf{Z}[i])$ for the points of $\operatorname{Spec} \mathbf{Z}$ and $\operatorname{Spec}(\mathbf{Z}[i])$ corresponding to the zero ideals. Obviously $^a\varphi(\omega') = \omega$ and $(^a\varphi)^{-1}(\{\omega\}) = \{\omega'\}$.

The other points of $\operatorname{Spec} \mathbf{Z}$ correspond to the prime numbers. By definition, $(^a\varphi)^{-1}(\{(p)\})$ is the set of prime ideals of $\mathbf{Z}[i]$ that divide p. As is well known, all such ideals are principal, and there are two of them if $p \equiv 1 \bmod 4$, and only one if $p = 2$ or $p \equiv 3 \bmod 4$. All of this can be pictured as in Figure 22.

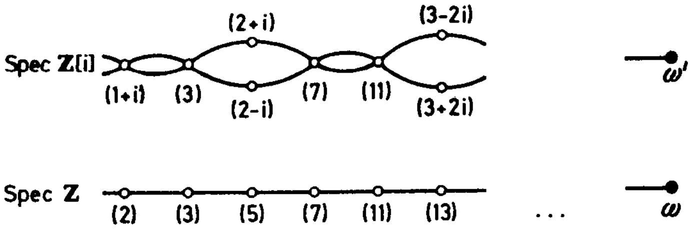

Figure 22. ${}^a\varphi$: Spec($\mathbf{Z}[i]$) $\to$ Spec $\mathbf{Z}$

We recommend the reader to work out the more complicated example of Spec($\mathbf{Z}[T]$), using the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}[T]$.

Example 4. Recall that a subset $S \subset A$ is a *multiplicative set* if it contains 1 and is closed under multiplication. For every multiplicative set, we can construct a ring of fractions A_S consisting of pairs (a, s) with $a \in A$ and $s \in S$, identified according to the rule

$$(a, s) = (a', s') \iff \exists\ s'' \in S \text{ such that } s''(as' - a's) = 0.$$

Algebraic operations are defined by the rules

$$(a, s) + (a', s') = (as' + a's, ss'),$$
$$(a, s)(a', s') = (aa', ss').$$

The reader will find a more detailed description of this construction in Atiyah and Macdonald [7], Chap. 3. From now on we write a/s for the pair (a, s). In particular, if S is the set $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A then A_S coincides with the local ring $A_\mathfrak{p}$ of A at a prime ideal (compare Chap. II, 1.1).

There is a map $\varphi\colon A \to A_S$ defined by $a \mapsto (a, 1)$, and hence a map

$${}^a\varphi\colon \operatorname{Spec}(A_S) \to \operatorname{Spec} A.$$

The reader can easily check that ${}^a\varphi$ is an inclusion, and that its image ${}^a\varphi(\operatorname{Spec}(A_S)) = U_S$ is the set of prime ideals of A disjoint from S. The inverse map $\psi\colon U_S \to \operatorname{Spec}(A_S)$ is of the form

$$\psi(\mathfrak{p}) = \mathfrak{p}A_S = \{x/s \mid x \in \mathfrak{p} \text{ and } s \in S\}.$$

In particular, if $f \in A$ and $S = \{f^n \mid n = 0, 1, \dots\}$ then A_S is denoted by A_f.

1.2. Properties of Points of Spec A

We can associate with each point $x \in \operatorname{Spec} A$ the field of fractions of the quotient ring by the corresponding prime ideal. This field is called the *residue field* at x and denoted by $k(x)$. Thus we have a homomorphism

$$A \to k(x),$$

whose kernel is the prime ideal we are denoting by x. We write $f(x)$ for the image of $f \in A$ under this homomorphism. If $A = k[X]$ is the coordinate ring of an affine variety X defined over an algebraically closed field k then $k(x) = k$, and for $f \in A$ the element $f(x) \in k(x)$ defined above is the value of f at x. In the general case each element $f \in A$ also defines a "function"

$$x \mapsto f(x) \in k(x)$$

on Spec A, but with the peculiarity that at different points x, it takes values in different sets. For example, when $A = \mathbf{Z}$, we can view any integer as a "function", whose value at (p) is an element of the field $\mathbf{F}_p = \mathbf{Z}/(p)$, and at (0) is an element of the rational number field $\mathbf{Q}$.

We now come up against one of the most serious points at which the "classical" geometric intuition turns out to be inapplicable in our more general situation. The point is that an element $f \in A$ is not always uniquely determined by the corresponding function on Spec A. For example, an element corresponds to the zero function if and only if it is contained in all prime ideals of A. These elements are very simple to characterise.

Proposition. *An element $f \in A$ is contained in every prime ideal of A if and only if it is nilpotent (that is, $f^n = 0$ for some n).*

Proof. See Appendix,[1] §6, Proposition 2 or Atiyah and Macdonald [7], Proposition 1.8.

Thus the inapplicability of the "functional" point of view in the general case is related to the presence of nilpotents in the ring. The set of all nilpotent elements of a ring A is an ideal, the *nilradical* of A.

For each point $x \in \operatorname{Spec} A$ there is a local ring $\mathcal{O}_x$, the local ring of A at the prime ideal x. For example, if $A = \mathbf{Z}$ and $x = (p)$ with p a prime number, then $\mathcal{O}_x$ is the ring of rational numbers a/b with denominator b coprime to p; if $x = (0)$ then $\mathcal{O}_x = \mathbf{Q}$.

This invariant of a point of Spec A allows us to extend to our general case a whole series of new geometric notions. For example, the definition of nonsingular points of a variety was related to purely algebraic properties of their local rings (Chap. II, 1.3). This prompts the following definition.

[1] Appendix refers to the Algebraic Appendix at the end of Book 1.