Irving E. Segal Ray A. Kunze

Integrals and Operators

Second Revised and Enlarged Edition

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Springer-Verlag
Berlin Heidelberg New York 1978

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First edition © 1968 by McGraw-Hill, Inc.

AMS Subject Classifications (1970): 22 B xx, 22 C 05, 22 D xx, 28-XX, 43-XX, 46-XX, 47-XX

ISBN 3-540-08323-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08323-5 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Segal, Irving Ezra. Integrals and operators (Grundlehren der mathematischen Wissenschaften; 228). Bibliography: p. Includes index. 1. Integrals, Generalized. 2. Functional analysis. I. Kunze, Ray Alden, 1928—joint author. II. Title. III. Series: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen; 228. QA312.S4. 1978. 515'.42. 77-16682

Printed in Great Britain
Printing and bookbinding: William Clowes & Sons Limited London, Beccles and Colchester
2141/3140-543210

PREFACE TO THE SECOND EDITION

Since publication of the First Edition several excellent treatments of advanced topics in analysis have appeared. However, the concentration and penetration of these treatises naturally require much in the way of technical preliminaries and new terminology and notation. There consequently remains a need for an introduction to some of these topics which would mesh with the material of the First Edition. Such an introduction could serve to exemplify the material further, while using it to shorten and simplify its presentation.

It seemed particularly important as well as practical to treat briefly but cogently some of the central parts of operator algebra and higher operator theory, as these are presently represented in book form only with a degree of specialization rather beyond the immediate needs or interests of many readers. Semigroup and perturbation theory provide connections with the theory of partial differential equations. C^*-algebras are important in harmonic analysis and the mathematical foundations of quantum mechanics. W^*-algebras (or von Neumann rings) provide an approach to the theory of multiplicity of the spectrum and some simple but key elements of the grammar of analysis, of use in group representation theory and elsewhere. The theory of the trace for operators on Hilbert space is both important in itself and a natural extension of earlier integration-theoretic ideas.

Accordingly, four chapters have been added, one dealing with each of the subjects indicated. These form a logical extension of the standpoint of the First Edition, and at the same time convey the fundamentals of subjects which are central for aspects of higher physical mathematics, group representation theory, and growing applications to analysis on manifolds.

The opportunity has been taken to correct errors, and terminological variations, as well as some expository lapses in the First Edition, which were kindly pointed out to us by conscientious readers. It is hoped the resulting volume will be useful to students and scientists in other fields who may be interested in a cultured overview of modern analysis and its logical structure which retains continuous connections with traditional real variable theory.

PREFACE TO THE FIRST EDITION

This book is intended as a first graduate course in contemporary real analysis. It is focused about integration theory, which we believe is appropriate. For a variety of reasons—in the interests of logic, flexibility, and curricular economy, among others—we have assumed that the reader or student is already familiar with the rudiments of modern mathematics (by this we mean the most elementary aspects of set theory, general topology, and algebra, as well as some exposure to rigorous analysis). These are not so much technical requirements—although basic concepts such as set, topological space, and uniform convergence are taken entirely for granted—as requirements of mathematical maturity and of understanding of the elementary grammar and language of modern mathematics. Assuming the adequate mathematical "aging" of the student, the book is quite self-contained. Results such as the Stone-Weierstrass theorem, the existence of a partition of unity, etc., are given full proofs rather than disposed of by reference to hypothesized preliminaries.

The aim of the book is primarily cultural, rather than *vocational; the* authors strive to expose the student to modern analytical thought and if possible to train him to think in such terms rather than to load him with all available information on the subject. Nevertheless, the book should represent a proper introduction to real analysis for students intending to concentrate in analysis, as well as a (possibly terminal) general course in the subject for those with other scientific interests. Indeed, thought cannot take place in a vacuum, and contrived illustrations of the theory have a way of turning out not to be as truly representative or interesting as illustrations of the actual usage of the theory for vital mathematical purposes. For this reason the book has been built around material of maximal current mathematical importance and depth, a technical mastery of which should go hand in hand with an appreciation of the general ideas.

The book is a revision, adaptation, and extension of lecture notes of courses in Integration Theory given at the University of Chicago and in Real Variable Theory at Massachusetts Institute of Technology by the first-named author. The first half of the book is suitable for a one-semester course in Lebesgue integration theory, including both abstract and classical real variable aspects. Its heart is a fresh presentation of the Daniell approach, which, combined with the use of general topological ideas, attains a high level of generality and completeness without burdening the student with heavy machinery or bulky technicalities. This material should provide a cultural experience for the student comparable to his first exposure to the calculus; indeed, the success of this theory against what appear initially as overwhelming scientific odds, and its broad applicability, render it one of the comparable intellectual achievements of mathematics. Many examples and a considerable variety of exercises, at all levels of difficulty, serve to illustrate the theory and to indicate the continuous transition between the concrete and abstract phases of integration theory. Theoretical ramifications which are secondary from the standpoint of the overall theoretical development, although frequently of considerable importance, are included among the more difficult exercises; the student is assisted with hints, and the arrangement is designed to encourage learning by students' investigations under the guidance of the instructor or by self-discovery. The exercises range in difficulty from easy ones which simply confirm an understanding of the text to relatively difficult ones, distinguished by a *, which in a controlled way, introduce the student to the beginnings of research. The * is also used to distinguish material (several sections and one chapter) which may be omitted without disturbing the main line of development.

The book as a whole is quite unified. Integration theory provides the main examples for the treatment of linear topological spaces and their duality. In the more structured situations provided by groups of transformations, new aspects of function theory arise from the consideration of invariant measures. The reducibility of commutative spectral analysis in Hilbert space to integration theory makes it natural, as well as economical, to develop spectral theory from this viewpoint. For these reasons the book may be used for the second half of a one-year course in Real Analysis, which merges naturally with the first half, provides basically new material of general importance, and yet serves at the same time to build on and provide a capstone for the student's earlier exposure to linear algebra and integration. The completion of such a course should provide the student with the key real-analytical background for work in other parts of modern mathematics; for more advanced work in analysis, whether of a more abstract or concrete variety; and for contemporary theoretical physics. We especially feel that the book takes the student rather quickly, but not too abruptly, to a good jumping-off place for the study of Fourier analysis, linear partial differential operators, the theory of group representations, operator algebras, and abstract probability theory.

Although there are now many quite competent treatments in textbook form of Lebesgue integration theory, as well as some on introductory functional analysis, we feel that none of these books achieves quite what this one is intended to do. With our treatment, it is possible to take the suitably prepared student in one year at a properly measured pace through basic contemporary real analysis, giving him the feel of the subject, a clear indication of its sweep, and an adequately detailed mastery of a number of central features. Our general viewpoint is partially in the direction of an earlier exposition by one of us on algebraic integration theory, i.e., toward the utilization of abstract integrative ideas (cf. References, p. 365); this has always been one of the long-term trends in mathematics, enforcing a type of consolidation which may be essential to prevent undue scientific complexity and bulk from imposing a crushing burden on the development of fresh ideas and methods. At the same time, as already indicated, the continuous linkage between the abstract theory and the concrete analytical situation has been everywhere insisted on—in the motivational material, in the examples, and in the exercises. We believe that this book is more likely to help cure "abstractionitis"—an unfortunate but not uncommon side effect of otherwise highly beneficial inoculations with modern mathematical ideas—than to cause it. We have made some technical innovations where they appeared useful to serve our central ideas, but have avoided them otherwise. Thus the treatment of "large" measure spaces is curtailed in the text, such primarily technical developments being outlined in the exercises; on the other hand, the uniform-space approach to the construction of invariant measures has been adopted.

As it has turned out, the book is fairly flexible from an instructional viewpoint. An independent short course, in the nature of an introduction to functional analysis and spectral theory in Hilbert space, may be given on the basis of the second half of the book, exclusive of Chapter 7, for students already familiar with the abstract Lebesgue integral.

We are much indebted to many colleagues and students for general advice and specific comments and for reducing the number of errors in the manuscript to what we hope is a superficial level. In particular, lengthy lists of corrections to draft manuscripts were supplied by Robert Kallman, Michael Weinless, and Alan Weinstein.

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CONTENTS

I

INTRODUCTION

1.1 GENERAL PRELIMINARIES

Before embarking on a serious study of a new subject, the intellectually prudent student will want to know why the subject is studied and what it relates to. Let us say that he accepts on faith the assurance that integration is significant, not only as a vital tool in analysis and as the culmination of the calculus, but also as an intrinsically beautiful and complete theory, in which elements of geometry and algebra, as well as analysis, are merged. Even so, his understanding of the subject will proceed more rapidly if he has some definite, if general, knowledge of what sort of thing it is and how it is related to the subjects he is already familiar with and if he sees why it has aroused such interest.

The Riemann integral that the reader is familiar with assigns to certain functions defined on certain sets in euclidean space a number called the integral of the function over the set. Both the function and the set must satisfy regularity conditions; it suffices, for example, if the function is continuous and the set is bounded and has a continuously differentiable boundary. In euclidean space there is defined a notion of length, area, or volume, depending on whether the dimension is 1, 2, or 3, or more generally, a notion of *n*-dimensional volume in *n*-space, which plays an important part in the formation of the integral. Now a common situation in analysis, as well as in quasi-mathematical subjects such as physics and probability, is that in which one is given a kind of notion of volume, or measure, for sufficiently smooth subsets of a basic set and wishes to assign to each sufficiently well-behaved function on the set a number that will be a sort of integral. The basic set may be a Riemannian manifold, or the phase space of a dynamical system, or the space of elementary events in a probabilistic system; the notion of volume may be derived from a given Riemannian metric, or may be determined by the condition that it be unchanged with the passage of time, or from probability considerations; it may be called area, mass, charge, or probability; and the integral in question may have the interpretation of volume, pressure, potential, or expectation. But these various situations are fundamentally similar in a certain way, or to use a more precise mathematical term, *isomorphic* with regard to certain aspects. The theory of integration studies the problem of assigning an integral to a function defined on a set that is endowed with a notion of "measure," with regard to those features that are independent of the origin of measure or the interpretation of the function. The "Lebesgue" integral, then, like any other integral, is a functional on a certain class of functions, which relates to a measure on the set over which the functions are defined.

The basic difference between the Riemann and Lebesgue integral is not so much in the measure—although historically a difference in the domains of definition of the relevant measures was taken very seriously—as in the fact that the Lebesgue integral extends the Riemann integral in a certain technically advantageous fashion, and yet is itself terminal, being incapable of any further extension of the same sort. In other terms, the Lebesgue integral applies to a class of functions that is maximal in a certain simple intrinsic sense and that includes the class of functions to which the Riemann integral applies. There are many other differences, such as the fact that the natural logical extension of the Lebesgue integral is to the case of functions on an abstract set, devoid of any topology, while the Riemann integral inherently refers to functions defined on a topological space; yet in essence the Lebesgue integral, in the historically original and still most important case of integration of functions on euclidean space with respect to euclidean measure, is both in a general and a mathematical sense a completion of the Riemann integral.

The basic problems in the theory of the Lebesgue integral are generally parallel to those in the theory of the Riemann integral, and yet proceed along completely different lines. First one must define the integral and prove its existence under usefully general conditions. Next one derives properties of the integral, but these are much more extensive in the Lebesgue case and include in particular the terminal feature described in the preceding paragraph. The primary general problems that remain are those of multiple integration and differentiation. The theory of differentiation is necessarily novel since it is logical, in general, to differentiate not a point function, as in the calculus, but a set function, the derivative being a point function; the reason for this will become apparent by the time this stage of the theory is reached. When the basic problems are covered, the theory has attained a certain logical completeness, but new problems emerge from the connections of the theory with other parts of mathematics. As in most substantial and living mathematics, there is a constant tension between the abstract and the concrete in the theory of integration. Although this may be as desirable as it is inevitable, it results in the impossibility of giving a single formulation of the subject that will be adaptable to the purposes of showing its evolution from the Riemann theory, of giving its logical basis in the most intelligible and elegant fashion, or of being conveniently applicable to Fourier analysis or to abstract functional analysis. The validity of all these purposes makes it undesirable, if not impractical or misleading, to present the subject for the first time in either a wholly abstract or wholly concrete fashion. Thus the theory will be treated in this book from a mixture of several points of view.

1.2 THE IDEA OF MEASURE

Traditionally, the Lebesgue integral referred to functions defined on euclidean space, or on a subset of it, like the familiar Riemann integral. As a result of both abstract and concrete impulses, however, mathematicians began looking into the question of whether the theory could not, in some essential parts at least, be carried over to more general situations. Even before Lebesgue, the Riemann-Stieltjes integral had been treated, showing that the Riemann theory applied to functions on euclidean space, but with a rather general type of measure, important in many applications. By the latter half of the thirties, after about a decade of activity along these lines, the basic theorems in the Lebesgue theory had achieved extensions to measures on perfectly arbitrary sets, although in a few cases (notably, differentiation theory) the extension did not have the full force of the original theory when applied to the euclidean case. Concurrently, these ideas had found important applications in probability, which found in the new theory a means of making explicit the notion of "random variable" which was basic in the subject. Since that time integration theory in abstract spaces has played a part in the spectral theory of linear operators (essentially, the extension of matrix theory to an infinite number of dimensions), in analysis on topological groups (a natural and widely applicable extension of classical Fourier analysis), and in many other diverse questions.

Thus the idea of abstract measure space has technical cogency, but it is also a very natural one from an intuitive geometrical point of view. To see how it arises in this way, consider the problem of formulating the concept of "measure" on a set S as an abstraction of the notions of volume, length, mass, probability, etc., mentioned above. The most obvious approach is to define this as a function that assigns to suitable subsets E of S a number $m(E)$, representing its measure and having the properties characteristic of measure. This then raises the primary questions of (*1*) which subsets are "suitable," and (*2*) what properties are characteristic. Now one property of the familiar examples of measure is that, so to speak, the whole is the sum of its parts, in terms of measure. More specifically, if A and B are disjoint sets for which the measure is defined, then their union has measure $m(A) + m(B)$. This seems intuitively, as well as technically, to be a reasonable assumption, but as might be expected, in itself it is not a sufficient basis for the development of a useful, interesting theory. For one thing, there is too little freedom in dealing with "suitable" sets if it is known only that the union of two disjoint suitable sets is again such. It does not seem too much to require, more stringently, that the union of any finite number of suitable sets be suitable, and even to add the further requirement that the difference of suitable sets be such. If this assumption is checked against the technical situation, it still seems quite moderate, and is satisfied (or can be arranged) in the familiar cases.

A collection $\mathcal{R}$ of subsets that is closed under the formation of the union and difference of any two subsets is generally called a *ring of sets*. In other words, $\mathcal{R}$ is a ring if whenever A and B are in $\mathcal{R}$, then $A \cup B$ and $A - B$ are again in $\mathcal{R}$. It is useful to justify the use of the term "ring" in the present context. The fact is that a ring $\mathcal{R}$ of sets is actually a ring, in the conventional algebraic sense, relative to certain operations, and conversely. Specifically, these operations are the intersection $A \cap B$ as the product, and the "symmetric difference" $A \ominus B = (A - B) \cup (B - A)$ as the sum. One verifies the algebraic identities that establish associativity, distributivity, etc., in the present connection, by taking a hypothetical element of the set on one side of the identity and showing that it is a member of the set on the other side. Hence, to show that a ring of sets $\mathcal{R}$ is a ring in the standard sense relative to these operations, it suffices to show that it is closed under the operations. For this it suffices to express the intersection and symmetric difference in terms of their union and (asymmetric) differences. The latter expression is given by the definition of the symmetric difference, while a corresponding expression for the intersection is

$$A \cap B = A - (A - B).$$

It should be noted here that the empty set is the unit for addition.

Conversely, if a collection $\mathcal{R}$ is a (standard type of) ring with respect to intersection as multiplication and the symmetric difference as addition, then