Initial-Boundary Value Problems for Nonlinear Parabolic Equations in Higher Dimensional Domains

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Preface

This book mainly deals with nonlinear parabolic equations and systems of second order in higher dimensional domains. We shall discuss several initial-boundary value problems for quasilinear, nonlinear parabolic equations and systems of second order equations with smooth coefficients and measurable coefficients.

In Chapter I, we first introduce some properties of solutions for parabolic equations of second order including the extremum principles, representation theorem and compactness principle of their solutions. By using the extremum principles, the uniqueness theorem of solutions for some initial-general boundary value problem is proved. The properties of solutions for parabolic equations will be used in the latter chapters.

In recent years, some initial-boundary value problems for nonlinear parabolic equations of second order with smooth coefficients were investigated by some mathematicians, but they only discussed the Dirichlet problem and initial-nonlinear regular oblique derivative problem for the equations. In Chapter II, we first introduce the solvability results of the above problems, and then consider the more general initial-nonlinear irregular oblique derivative problem.

In Chapters III and IV, not only several initial-boundary value problems for nonlinear nondivergent parabolic equations of second order with measurable coefficients, i.e. Cordes coefficients, but also some initial-boundary value problems for nonlinear nondivergent parabolic systems of second order equations with measurable coefficients are investigated, which cannot be found in other books published. Here we first give a priori estimates of solutions for the above initial-boundary value problems and then prove their solvability by the estimates of solutions and the method of parameter extension or the Leray-Schauder theorem.

In Chapter V, we discuss some initial-boundary value problems for linear and quasilinear parabolic equations of second order with other measurable coefficients, i.e. VMO coefficients. We give a priori estimates of solutions for the above problems, and prove the uniqueness and existence of solutions for the problems, here we mainly construct the foundational solution of the corresponding initial-boundary value problems for linear parabolic equation of second order.

There are two characteristics of this book: one is that parabolic equations are discussed in the nonlinear case, and the boundary conditions include the irregular oblique derivative case, another is that boundary value problems are almost considered in the case of multiply connected domains and several methods are used. We mention that the methods in this book can be used to discuss the corresponding boundary value problems for nonlinear elliptic equations in higher dimensional domains, and some moving boundary problems in filtrations, gas dynamics, elastico-plastic mechanics can be handled by using the results as stated in this book.

The great majority of the contents originates in investigations of the authors and his cooperative colleagues, and many results are published here for the first time. After reading the volume, it can be seen that many questions remain for further investigations.

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Contents

Chapter I
Properties of Solutions for Parabolic Equations of Second Order

In this chapter, we mainly introduce some properties of solutions for linear parabolic equations of second order including the extremum principles, representation theorem and compactness principle of their solutions. By using the extremum principles, the uniqueness and stability of solutions for some initial-boundary value problems are verified. Besides the existence of solutions of the Dirichlet problem for linear parabolic equations is proved. The properties of solutions for parabolic equations will be used in the following chapters.

1. Conditions of Linear and Nonlinear Parabolic Equations of Second Order

Let Ω be a bounded domain in $\mathbf{R}^N$ with the boundary $\partial\Omega \in C^2_\mu$ $(0 < \mu < 1)$ and $Q = \Omega \times I$ be a cylinder, here $I = \{0 < t \le T\}$, T is a positive constant, and $\partial Q = \partial Q_1 \cup \partial Q_2$ is called the parabolic boundary of Q, in which $\partial Q_1, \partial Q_2$ are the bottom $\{x \in \Omega, t = 0\}$ and the lateral boundary $\{x \in \partial\Omega,\ t \in \bar{I}\}$ of the domain Q respectively.

First of all, we consider the linear partial differential equation of second order

$$\mathcal{L}u = \sum_{i,j=1}^N a_{ij}u_{x_ix_j} + \sum_{i=1}^N b_i u_{x_i} + cu - u_t = f \ \text{ in } Q, \tag{1.1}$$

where the coefficients $a_{ij} = a_{ij}(x,t)$, $b_i = b_i(x,t)(i,j = 1,\cdots,N)$, $c = c(x,t)$, $f(x,t)$ are known continuous functions in Q. The condition of uniformly parabolic type for equation (1.1) is that the following inequality holds

$$q_0 \sum_{j=1}^N |\xi_j|^2 \le \sum_{i,j=1}^N a_{ij}\xi_i\xi_j \le q_0^{-1} \sum_{j=1}^N |\xi_j|^2 \ \text{ in } Q, \tag{1.2}$$

in which $q_0 (0 < q_0 < 1)$ is a constant. If $a_{ij}, b_i(i,j = 1,\cdots,N)$, c, f satisfy

$$||\eta||_{C^{1,0}_{\alpha,\alpha/2}(Q^*)} \le k_0, \ \ \eta = a_{ij}, b_i(i,j = 1,\cdots,N), c, f, \tag{1.3}$$

where Q^* is any closed subdomain in Q, $\alpha (0 < \alpha < 1), k_0$ are non-negative constants and

$$||a||_{C^{1,0}(Q^*)} = ||a||_{C^{0,0}(Q^*)} + \sum_{i=1}^{N} ||a_{x_i}||_{C^{0,0}(Q^*)},$$

$$||a||_{C^{1,0}_{\alpha,\alpha/2}(Q^*)} = ||a||_{C^{0,0}_{\alpha,\alpha/2}(Q^*)} + \sum_{i=1}^{N} ||a_{x_i}||_{C^{0,0}_{\alpha,\alpha/2}(Q^*)},$$

$$||a||_{C_{\alpha,\alpha/2}(Q^*)} = ||a||_{C^{0,0}(Q^*)} + ||a||_{H_{\alpha,\alpha/2}(Q^*)}$$

$$= \max_{Q^*} |a| + \max_{(x,t)\neq(y,\tau)\in Q^*} \frac{|a(x,t) - a(y,\tau)|}{|x-y|^{\alpha} + |t-\tau|^{\alpha/2}},$$

then we say that equation (1.1) satisfies Condition C_0. In this case, if a function $u(x,t) \in C^{2,1}_{\alpha,\alpha/2}(Q)$ satisfies equation (1.1) for every point $(x,t) \in Q$, then the function is called a classical solution of (1.1) in Q.

Secondly, we consider the nonlinear parabolic equation of second order

$$F(x,t,u,D_x u, D_x^2 u) - u_t = 0 \text{ in } Q, \tag{1.4}$$

namely

$$\sum_{i,j=1}^{N} a_{ij} u_{x_i x_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu - u_t = f \text{ in } Q, \tag{1.5}$$

where $D_x u = (u_{x_i}), D_x^2 u = (u_{x_i x_j})$, and

$$a_{ij} = \int_0^1 F_{\tau r_{ij}}(x,t,u,p,\tau r)\mathrm{d}\tau, \ b_i = \int_0^1 F_{\tau p_i}(x,t,u,\tau p, 0)\mathrm{d}\tau,$$

$$c = \int_0^1 F_{\tau u}(x,t,\tau u, 0, 0)\mathrm{d}\tau, \ f = -F(x,t,0,0,0),$$

$$p = D_x u, \ r = D_x^2 u, \ p_i = \frac{\partial u}{\partial x_i}, \ r_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \ i,j = 1, \cdots, N.$$

Suppose that (1.4) (or (1.5)) satisfies Condition C, i.e. for arbitrary functions $u(x,t)$, $u^1(x,t)$, $u^2(x,t) \in C^{1,0}_{\alpha,\alpha/2}(\overline{\Omega}) \cap \tilde{W}_2^{2,1}(Q)$, $F(x,t,u,D_x u, D_x^2 u)$ satisfies the condition

$$F(x,t,u^1,D_x u^1, D_x^2 u^1) - F(x,t,u^2,D_x u^2, D_x^2 u^2)$$
$$= \sum_{i,j=1}^{N} \tilde{a}_{ij} u_{x_i x_j} + \sum_{i=1}^{N} \tilde{b}_i u_{x_i} + \tilde{c} u, \tag{1.6}$$

where $u = u^1 - u^2$ and

$$\tilde{a}_{ij} = \int_0^1 F_{u_{x_ix_j}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau, \ \ \tilde{b}_i = \int_0^1 F_{u_{x_i}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau,$$

$$\tilde{c} = \int_0^1 F_u(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau, \ \ \tilde{u}=u^2+\tau(u^1-u^2), \ \ \tilde{p}=D_x\tilde{u}, \ \ \tilde{r}=D_x^2\tilde{u},$$

$$||u||_{\tilde{W}_2^{2,1}(Q)} = (\int_Q [u^2 + \sum_{i=1}^N u_{x_i}^2 + \sum_{i,j=1}^N u_{x_ix_j}^2 + u_t^2]\mathrm{d}x\mathrm{d}t)^{1/2},$$

and $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, f$ satisfy the condition

$$q_0\sum_{j=1}^N|\xi_j|^2 \le \sum_{i,j=1}^N \tilde{a}_{ij}\xi_i\xi_j \le q_0^{-1}\sum_{j=1}^N|\xi_j|^2, \ \ 0<q_0<1, \tag{1.7}$$

$$\begin{gathered}\frac{\sup_Q \sum_{i,j=1}^N \tilde{a}_{ij}^2}{\inf_Q[\sum_{i=1}^N \tilde{a}_{ii}]^2} \le q_1 < \frac{1}{N-1/2}, \ \ L_p[f,\overline{Q}] \le k_1, \\ |\tilde{a}_{ij}| \le k_0, \ |\tilde{b}_i| \le k_0, \ i,j=1,\cdots,N, \ |\tilde{c}| \le k_0,\end{gathered} \tag{1.8}$$

where $q_0, q_1, k_0, k_1, p\,(> N+2)$ are non-negative constants. Moreover, for almost every point $(x,t) \in Q$ and $D_x^2u \in \mathbf{R}^{N(N+1)/2}$, $\tilde{a}_{ij}(x,t,u,D_xu,\ D_x^2u), \tilde{b}_i(x,t,u,D_xu), \tilde{c}(x,t,u)$ are continuous in $u \in \mathbf{R}, D_xu \in \mathbf{R}^N$. If the last two conditions in (1.8) are replaced by

$$\begin{gathered}L_p[\tilde{b}_i,\overline{Q}] \le k_0, \ i=1,\cdots,N, \\ L_p[\tilde{c},\overline{Q}] \le k_0, \ p > N+2, \ \sup_Q \tilde{c} < \infty,\end{gathered} \tag{1.9}$$

then Condition C will be called Condition C'. If the first condition in (1.8) is replaced by

$$\sup_Q \frac{\sum_{i,j=1}^N \tilde{a}_{ij}^2}{[\sum_{i=1}^N \tilde{a}_{ii}]^2} \le q_1 < \frac{1}{N-\tilde{\mu}^2}, \ \tilde{\mu} = \frac{\inf_Q \sum_{i=1}^N \tilde{a}_{ii}}{\sup_Q \sum_{i=1}^N \tilde{a}_{ii}}, \tag{1.10}$$

then Condition C' will be called Condition C''. It is not difficult to derive the following relation

$$\text{Condition } C \ \subset \text{Condition } C' \subset \text{Condition } C''.$$

The so-called Dirichlet problem (Problem D), initial-Neumann problem (Problem N) and the initial-regular oblique derivative problem (Problem O) of equation (1.1), i.e. to find a continuous solution

$u = u(x,t) \in C^{1,0}_{\beta,\beta/2}(\overline{Q}) \cap \tilde{W}^{2,1}_2(Q)$ of (1.1) satisfying the initial condition

$$u(x,0) = g(x), \quad x \in \Omega, \tag{1.11}$$

and the boundary conditions

$$u(x,t) = r(x,t), \quad (x,t) \in \partial Q_2 \,(\text{Problem } D), \tag{1.12}$$

$$\frac{\partial u(x,t)}{\partial \vec{n}} = \tau(x,t), \quad (x,t) \in \partial Q_2 \,(\text{Problem } N), \tag{1.13}$$

$$\frac{\partial u(x,t)}{\partial \vec{\nu}} + \sigma(x,t)u = \tau(x,t), \quad (x,t) \in \partial Q_2 \,(\text{Problem } O), \tag{1.14}$$

respectively, in which $\vec{\nu}$ and $\vec{n}$ are the unit vector and unit outward normal at every point $(x,t) \in \partial Q_2$ respectively, and $g(x)$ in $\bar{\Omega}$, $r(x,t)$, $\sigma(x,t)$, $\tau(x,t)$, $\cos(\vec{\nu},\vec{n}) > 0$ on ∂Q_2 satisfy the conditions

$$\begin{aligned} &C^2_\alpha[g(x),\bar{\Omega}] \le k_2, \quad C^{1,1}_{\alpha,\alpha/2}[r(x,t),\partial Q_2] \le k_2, \\ &\sigma(x,t) \ge 0 \text{ on } Q, \quad C^{1,1}_{\alpha,\alpha/2}[\tau(x,t),\partial Q_2] \le k_2, \\ &C^{1,1}_{\alpha,\alpha/2}[\eta(x,t),\partial Q_2] \le k_0, \quad \eta = \sigma, \cos(\vec{\nu},\vec{n}), \end{aligned} \tag{1.15}$$

where $\alpha\,(0 < \alpha < 1), k_0, k_2$ are non-negative constants. There is no harm in assuming that $\sigma(z) > 0$ on ∂Q_2 in (1.14), because otherwise we can find a solution $\Psi(x,t)$ of the equation

$$\Delta u - u_t = 0, \quad \text{i.e. } \sum_{i=1}^N u_{x_i x_i} - u_t = 0 \text{ in } Q, \tag{1.16}$$

satisfying the boundary condition $\Psi(x,t) = 1$ on ∂Q, then the $v(x,t) = u(x,t)/\Psi(x,t)$ is a solution of the equation

$$\sum_{i,j=1}^N a_{ij} v_{x_i x_j} + \sum_{i=1}^N \tilde{b}_i v_{x_i} + \tilde{c} v - v_t = \tilde{f} \text{ in } Q \tag{1.17}$$

satisfying the initial-boundary condition (1.11) and

$$\frac{\partial v(x,t)}{\partial \vec{\nu}} + \tilde{\sigma}(x,t)v = \tilde{\tau}(x,t), \quad (x,t) \in \partial Q_2, \tag{1.18}$$

in which

$$\tilde{b}_i = b_i + \sum_{j=1}^{N}[a_{ij}(\ln\Psi)_{x_j} + a_{ji}(\ln\Psi)_{x_j}],\quad \tilde{f} = f/\Psi,$$

$$\tilde{c} = c + \sum_{i,j=1}^{N} a_{ij}\frac{\Psi_{x_ix_j}}{\Psi} + \sum_{i=1}^{N} b_i(\ln\Psi)_{x_i} - (\ln\Psi)_t \text{ in } Q,$$

$$\tilde{\sigma}(x,t) = \frac{\partial[\ln\Psi(x,t)]}{\partial\vec{\nu}} + \sigma(x,t) > 0,\quad \tilde{\tau}(x,t) = \frac{\tau(x,t)}{\Psi(x,t)} \text{ on } \partial Q_2,$$

here $\partial[\ln\Psi(x,t)]/\partial\vec{\nu} > 0$ on ∂Q_2 can be derived by Lemma 2.1 and Theorem 2.4 below.

The solution $u(x,t)$ of equation (1.5) with Condition C is indicated a function $u(x,t) \in C^{1,0}_{\alpha,\alpha/2}(Q) \cap \tilde{W}^{2,1}_2(Q^*)$, i.e. u, u_{x_i} $(i = 1,\cdots,N)$ $\in C^{0,0}_{\alpha,\alpha/2}(Q)$ $(0 < \alpha < 1)$, $u_{x_ix_j}$ $(i,j = 1,\cdots,N), u_t \in L_2(Q^*)$, and $u(x,t)$ satisfies equation (1.5) for almost every point $(x,t) \in Q$, in which $C^{0,0}(Q) = C(Q)$ and Q^* is any closed subset in the domain Q. In this case, the solution of (1.5) is called a generalized solution in Q.

If the linear equation (1.1) with Condition C_0 satisfies $c(x,t) \le 0$, $f(x,t) \ge 0$ and $c(x,t) \le 0$, $f(x,t) \le 0$ in Q, then we say that equation (1.1) satisfies Condition C_0^+ and Condition C_0^- respectively. Besides, equation (1.1) with Condition C and the conditions $c(x,t) \le 0$, $f(x,t) \ge 0$ for almost every point $(x,t) \in Q$ will be denoted by Condition C^+, and Condition C and the conditions $c(x,t) \le 0$, $f(x,t) \le 0$ for almost every point $(x,t) \in Q$ will be denoted by Condition C^-, and the conditions corresponding to Condition C'' will be called Condition C''^+ and Condition C''^-. In the next section, we shall prove the extremum principles of solutions for equation (1.1) with above smooth coefficients and measurable coefficients.

2. Extremum Principles of Solutions for Parabolic Equations of Second Order

We first prove some extremum principles of solutions for equation (1.1) with Condition C_0, and then verify some extremum principles of solutions for (1.1) with Condition C.

2.1 Extremum principles of solutions for parabolic equations with continuous coefficients

Lemma 2.1 *Suppose that equation (1.1) satisfies Condition C_0^+, and $u(x,t)$ is a continuous solution of (1.1) in $\overline{Q}$. If $u(x,t)$ attains the non-negative maximum at a point $P^0=(x^0,t^0)\in\partial Q_2$ and $u(x,t) < u(x^0,t^0)$ for $(x,t)\in Q$, then*

$$\varliminf_{P\to P^0}\frac{u(P^0)-u(P)}{r(P^0,P)}>0, \tag{2.1}$$

where $r(P^0,P)=[\sum_{i=1}^N(x_i-x_i^0)^2+|t-t^0|]^{1/2}$, $P=(x,t)$ approaches P^0 along a direction $\vec{l}$, $\cos(\vec{l},\vec{n})>0$, $\vec{n}$ is the outward normal at P^0 on ∂Q_2.

Proof We make an inner tangent ball S in Q with the tangent point at P^0, whose center is the point $P^1=(x^1,t^0)$ and its radius is $R>0$, and denote by ∂S the boundary of the ball S. Consider the auxiliary function

$$V(P)=\mathrm{e}^{-\alpha[r^2+(t-t^0)^2]}-\mathrm{e}^{-\alpha R^2}, \tag{2.2}$$

where $r=|x-x^1|=[\sum_{i=1}^N(x_i-x_i^1)^2]^{1/2}$, α is an undetermined positive constant. It is clear that $V(P)>0$ in S, and $V(P)=0$ on ∂S. Through the direct calculation, we obtain

$$\begin{aligned}\mathcal{L}V&=\sum_{i,j=1}^N a_{ij}V_{x_ix_j}+\sum_{i=1}^N b_iV_{x_i}+cV-V_t\\&=cV+\mathrm{e}^{-\alpha[r^2+(t-t^0)^2]}[\alpha^2\sum_{i,j=1}^N a_{ij}(x_i-x_i^1)(x_j-x_j^1)\\&\quad-2\alpha\sum_{i=1}^N(a_{ii}+b_i(x_i-x_i^1))+2\alpha(t-t^0)].\end{aligned}$$

Setting $S_1=\{|x-x^1|^2+|t-t^0|^2\le R^2,\ |x-x^1|^2\ge R^2/4\}$, and choosing that α is large enough such that $\mathcal{L}V>0$ in S_1, we make an auxiliary function

$$W(P)=\varepsilon V(P)+u(P)-u(P^0).$$

Selecting the sufficiently small positive number ε such that $W(P)<0$ on $S\cap\{|x-x^1|^2=R^2/4\}$ and $W(P)\le 0$ on ∂S, we can conclude

$W(P) \le 0$ in S_1. Otherwise, there exists an inner point P^2 of S_1 so that $W(P)$ attains the positive maximum at P^2. According to the property of maximum point for the function of several variables, we have

$$W \ge 0,\ cW \le 0,\ \ W_{x_i} = 0,\ i = 1, \cdots, N,$$

$$W_t = 0,\ \ \sum_{i,j=1}^{N} a_{ij} W_{x_i x_j} \le 0 \ \text{ at } \ P^2.$$

We can see $\mathcal{L}W \le 0$ at P^2. However by the definition of $W(P)$ we have

$$\mathcal{L}W = \varepsilon \mathcal{L}V(P) + \mathcal{L}u(P) - \mathcal{L}u(P^0) > 0 \ \text{ at } \ P^2.$$

The contradiction shows that $W(P) \le 0$ for $P \in S_1$. Noting that $V(P^0) = 0$, hence

$$u(P^0) - u(P) \ge -\varepsilon[V(P^0) - V(P)] \ \text{ for } \ P \in S_1.$$

Thus when P approaches P^0 along the outward normal $\vec{n}$, we get

$$\varliminf_{P \to P^0} \frac{u(P^0) - u(P)}{r(P^0, P)} \ge -\varepsilon \frac{\partial V}{\partial \vec{n}} = 2\varepsilon \alpha R e^{-\alpha R^2} > 0. \qquad (2.3)$$

Noting the condition $\varliminf_{P(\in \vec{s}) \to P^0} (u(P^0) - u(P))/r(P^0, P) \ge 0$, $\cos(\vec{l}, \vec{n}) > 0$, $\cos(\vec{l}, \vec{s}) > 0$ at P^0, where $\vec{s}$ is a tangent vector of ∂Q_2 at P^0, from (2.3), it follows the inequality

$$\varliminf_{P(\in \vec{l}) \to P^0} \frac{u(P^0) - u(P)}{r(P^0, P)} \ge \cos(\vec{l}, \vec{n}) \varliminf_{P(\in \vec{n}) \to P^0} \frac{u(P^0) - u(P)}{r(P^0, P)}$$

$$+ \cos(\vec{l}, \vec{s}) \varliminf_{P(\in \vec{s}) \to P^0} \frac{u(P^0) - u(P)}{r(P^0, P)} > 0,$$

i.e. (2.1) holds.

Lemma 2.2 *Suppose that equation* (1.1) *satisfies Condition* C_0^-, *and* $u(x,t)$ *is a continuous solution of* (1.1) *in* $\overline{Q}$. *If* $u(x,t)$ *attains the non-positive minimum at a point* $P^0 = (x^0, t^0) \in \partial Q_2$ *and* $u(x,t) > u(x^0, t^0)$ *for* $(x,t) \in Q$, *then*

$$\varlimsup_{P \to P^0} \frac{u(P^0) - u(P)}{r(P^0, P)} < 0, \qquad (2.4)$$

where $P = (x,t)$ *approaches* P^0 *along a direction* $\vec{l}$, $\cos(\vec{l}, \vec{n}) > 0$, $\vec{n}$ *and* $r(P^0, P)$ *are defined as in* (2.1).

Proof Put $v(P) = -u(P)$, it is not difficult to see that $v(P)$ possesses the property of $u(P)$ in Lemma 2.1. By using (2.1), we obtain

$$\overline{\lim_{P\to P^0}} \frac{u(P^0) - u(P)}{r(P^0, P)} = -\lim_{\overline{P\to P^0}} \frac{v(P^0) - v(P)}{r(P^0, P)} < 0. \qquad (2.5)$$

On the basis of Lemmas 2.1 and 2.2, it is easy to derive the following corollary.

Corollary 2.3 *Let S be a ball in $\overline{Q}$.*

(1) *If $u(x,t)$ is a continuous solution of equation* (1.1) *with Condition C_0^+ in $\overline{Q}$, and $u(x,t)$ does not take the non-negative maximum at the inner point of S, then $u(x,t)$ attains the non-negative maximum at the south pole or north pole of ∂S.*

(2) *If $u(x,t)$ is a continuous solution of equation* (1.1) *with Condition C_0^- in $\overline{Q}$, and $u(x,t)$ does not take the non-positive minimum at the inner point of S, then $u(x,t)$ attains the non-positive minimum at the south pole or north pole of ∂S.*

Next, we shall prove the maximum principle and minimum principle of solutions for equation (1.1) with some conditions.

Theorem 2.4 *Suppose that equation* (1.1) *satisfies Condition C_0^+ or Condition C_0^-, and $u(x,t)$ is a continuous solution of* (1.1) *in $\overline{Q}$. If $u(x,t)$ attains the non-negative maximum or non-positive minimum at an inner point $P^0 = (x^0, t^0)$ of $\overline{Q}$ respectively, then*

$$u(x,t) = u(x^0,t^0) = u(P^0) \text{ in } \overline{Q_{t^0}} = \{(x,t) \mid x \in \overline{Q}, 0 \le t \le t^0\}.$$

Proof We first prove that if $u(x,t) = u(P)$ takes the non-negative maximum $u(P^0)$ at an inner point (x^*, t^*) of Q_{t^0}, then $u(P) = u(P^0)$ on the point set $Q_* = Q_{t^*} \cap \{t = t^*\}$, the t-coordinates of which are equal to t^*. Otherwise, there exists a point $P' = (x', t^*) \in Q_*$ such that $u(P') < u(P^0)$. Denote $E_0 = \{(x,t^*) \mid u(x,t^*) = u(P^0), (x,t^*) \in Q_{t^*}\}$, it is obvious that E_0 is a closed set in Q_*. Thus it is not difficult to find a ball on $t = t^*$ with the center in Q_*, which is tangent to an inner point $P^1 = (x^1, t^*)$ in Q_* and $P^1 \in E_0$ such that $u(P) < u(P^1)$ in the ball. By means of Lemma 2.1, we can see that $\partial u/\partial \vec{n} > 0$ at P^1. On the other hand, we have $\partial u/\partial x_i = 0\,(i = 1, \cdots, N)$ at P^1. This contradiction proves that $u(P) = u(P^1)$ on Q_*.

Denote $F = \{(x,t) \mid u(P) = u(P^0), P = (x,t) \in \overline{Q_{t^0}}\}$, obviously F is a closed set. Suppose that $F \neq \overline{Q_{t^0}}$, then there exists a point $P' = (x', t') \in \overline{Q_{t^0}}$ and $P' \notin F$ such that a line segment with the end points P' and P^0 is included in Q_{t^0}, and denote by $P^1 = (x^1, t^1)$ the first intersection of the line segment and F. Making a cylinder G_1 with the height d, center of whose upper bottom is P^1 and whose generator is parallel to t-axis. Setting a ball $S = \{(x-x^1)^2 + (t-t^1 + d+h)^2 \leq (d+h)^2\}$, $G_2 = G_1 \cap S$, and $\Gamma_1 = G_2 \cap \partial S$, $\Gamma_2 = \partial G_2 \backslash \Gamma_1$, we introduce a function

$$V(P) = |x - x^1|^2 + (t - t^1 + d + h)^2 - (d+h)^2. \qquad (2.6)$$

It is clear that $V(P) < 0$ in S, and $V(P) = 0$ on ∂S. Provided that the positive number h is large enough and positive number d is sufficiently small, we get

$$\begin{aligned}\mathcal{L}V =& 2\sum_{i=1}^{N}[a_{ii}+b_i(x_i-x_i^1)]+c[|x-x^1|^2+(t-t^1+d+h)^2-(d+h)^2]\\ &-2(t-t^1+d+h) = 2\sum_{i=1}^{N}[a_{ii}+b_i(x_i-x_i^1)]+c[|x-x^1|^2+(t-t^1+d)^2\\ &-d^2] - 2(t-t^1+d) - 2h[1 - c(t-t^1+d) + cd] < 0 \ \text{ in } G_2.\end{aligned} \qquad (2.7)$$

Moreover, we consider the auxiliary function

$$W(P) = -\varepsilon V(P) + u(P),$$

where ε is a sufficiently small positive constant. Obviously $W(P)$ does not take the maximum in G_2, because $\mathcal{L}W = -\varepsilon\mathcal{L}V + \mathcal{L}u > 0$ by the definition of $W(P)$, and $\mathcal{L}W(P) \leq 0$ at the maximum point of $W(P)$. In addition, noting that ε is small enough such that

$$W(P) = -\varepsilon V(P) + u(P) < u(P^0) = W(P^1) \ \text{ on } \Gamma_2, \qquad (2.8)$$

this shows that $W(P)$ does not attain the maximum on Γ_2. Thus the maximum of $W(P)$ on G_2 only attains at the point $P^1 \in \Gamma_1$, hence we get

$$W_t = -\varepsilon V_t + u_t = 0 \ \text{ at } P^1,$$

however $V_t = 2(t - t_1 + d + h) > 0$ in G_2. Consequently $u_t > 0$, it follows that $\mathcal{L}u \leq cu - u_t < 0$ at P^1. This contradicts the hypothesis as before. Therefore $u(z,t) = u(P^0)$ in $\overline{Q_{t^0}}$.

By the similar method we can prove that if $u(x,t)$ attains the non-positive minimum at an inner point $P^0 = (x^0, t^0)$ in Q, then $u(x,t) = u(P^0)$ in $\overline{Q_{t^0}}$.

2.2 Extremum principles of solutions for parabolic equations with measurable coefficients

In the following, we consider equation (1.1) with Condition C, Condition C^+ or Condition C^- as stated in Section 1.

Theorem 2.5 *Let equation* (1.1) *satisfy Condition* C'^+ *and* $u(x,t)$ *be a continuous solution in* $\overline{Q}$. *Then the positive maximum of* $u(x,t)$ *attains at a point on* ∂Q.

To prove the above theorem, we first show a lemma.

Lemma 2.6 *If equation* (1.1), *i.e.*

$$\sum_{i,j=1}^{N} a_{ij}u_{x_ix_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu - u_t = f \tag{2.9}$$

satisfies Condition C'^+, *then the continuous solution* $u(x,t)$ *of* (2.9) *in* $\overline{Q}$ *attains its maximum or minimum at a point on* ∂Q.

Proof It is sufficient to prove the statement in Lemma 2.6 for any cylinder S, there is no harm in assuming that $S = \{(x,t) \,|\, |x| \le 1,\ 0 \le t \le 1\}$. We choose the sequence of equations

$$\sum_{i,j=1}^{N} a_{ij}^n u_{x_ix_j} + \sum_{i=1}^{N} b_i^n u_{x_i} + c^n u - u_t = f^n,\ \ n = 1, 2, \cdots, \tag{2.10}$$

where coefficients $a_{i,j}^n(x,t)$, $b_i^n\,(i,j = 1,\cdots,N)$, c^n, f^n are continuously differentiable in S and satisfy Condition C_0^+, and $a_{i,j}^n(x,t)$, $b_i^n\,(i,j = 1,\cdots,N)$, c^n, f^n converge to $a_{i,j}(x,t)$, $b_i\,(i,j = 1,\cdots,N)$, c, f according to the norm $L_p(\overline{Q})$ $(p > N+2)$ as $n \to \infty$ respectively. The existence of coefficients $a_{i,j}^n(x,t)$, b_i^n $(i,j = 1,\cdots,N)$, c^n, f^n can be seen in [81]. Next, we can find a solution $u_n(x,t)$ of the Dirichlet problem for (2.10) with the boundary condition

$$u_n(x,t) = u(x,t) \ \text{ on } \partial S = \partial S_1 \cup \partial S_2, \tag{2.11}$$

where $\partial S_1 = \{(x,t) \,|\, |x| \le 1, t = 0\}$, $\partial S_2 = \{(x,t) \,|\, |x| = 1, 0 \le t \le 1\}$ and every function of $\{u_n(x,t)\}$ satisfies the estimates similar