Graduate Texts in Mathematics

Béla Bollobás

Modern Graph Theory

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Modern Graph Theory

With 118 Figures

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To Gabriella

As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as any human undertaking pursues certain objects, so also mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.

David Hilbert, *Mathematical Problems*,
International Congress of Mathematicians,
Paris, 1900.

Apologia

This book has grown out of *Graph Theory – An Introductory Course* (GT), a book I wrote about twenty years ago. Although I am still happy to recommend GT for a fairly fast-paced introduction to the basic results of graph theory, in the light of the developments in the past twenty years it seemed desirable to write a more substantial introduction to graph theory, rather than just a slightly changed new edition.

In addition to the classical results of the subject from GT, amounting to about 40% of the material, this book contains many beautiful recent results, and also explores some of the exciting connections with other branches of mathematics that have come to the fore over the last two decades. Among the new results we discuss in detail are: Szemerédi's Regularity Lemma and its use, Shelah's extension of the Hales-Jewett Theorem, the results of Galvin and Thomassen on list colourings, the Perfect Graph Theorem of Lovász and Fulkerson, and the precise description of the phase transition in the random graph process, extending the classical theorems of Erdős and Rényi. One whole field that has been brought into the light in recent years concerns the interplay between electrical networks, random walks on graphs, and the rapid mixing of Markov chains. Another important connection we present is between the Tutte polynomial of a graph, the partition functions of theoretical physics, and the powerful new knot polynomials.

The deepening and broadening of the subject indicated by all the developments mentioned above is evidence that graph theory has reached a point where it should be treated on a par with all the well-established disciplines of pure mathematics. The time has surely now arrived when a rigorous and challenging course on the subject should be taught in every mathematics department. Another reason why graph theory demands prominence in a mathematics curriculum is its status as that branch of pure mathematics which is closest to computer science. This proximity enriches both disciplines: not only is graph theory fundamental to theoretical computer science, but problems arising in computer science and other areas of application greatly influence the direction taken by graph theory. In this book we shall not stress applications: our treatment of graph theory will be as an exciting branch of pure mathematics, full of elegant and innovative ideas.

Graph theory, more than any other branch of mathematics, feeds on problems. There are a great many significant open problems which arise naturally in the subject: many of these are simple to state and look innocent but are proving to be surprisingly hard to resolve. It is no coincidence that Paul Erdős, the greatest problem-poser the world has ever seen, devoted much of his time to graph theory. This amazing wealth of open problems is mostly a blessing, but also, to some extent, a curse. A blessing, because there is a constant flow of exciting problems stimulating the development of the subject: a curse, because people can be misled into working on shallow or dead-end problems which, while bearing a superficial resemblence to important problems, do not really advance the subject.

In contrast to most traditional branches of mathematics, for a thorough grounding in graph theory, absorbing the results and proofs is only half of the battle. It is rare that a genuine problem in graph theory can be solved by simply applying an existing theorem, either from graph theory or from outside. More typically, solving a problem requires a "bare hands" argument together with a known result with a new twist. More often than not, it turns out that none of the existing high-powered machinery of mathematics is of any help to us, and nevertheless a solution emerges. The reader of this book will be exposed to many examples of this phenomenon, both in the proofs presented in the text and in the exercises. Needless to say, in graph theory we are just as happy to have powerful tools at our disposal as in any other branch of mathematics, but our main aim is to solve the substantial problems of the subject, rather than to build machinery for its own sake.

Hopefully, the reader will appreciate the beauty and significance of the major results and their proofs in this book. However, tackling and solving a great many challenging exercises is an equally vital part of the process of becoming a graph theorist. To this end, the book contains an unusually large number of exercises: well over 600 in total. No reader is expected to attempt them all, but in order to really benefit from the book, the reader is strongly advised to think about a fair proportion of them. Although some of the exercises are straightforward, most of them are substantial, and some will stretch even the most able reader.

Outside pure mathematics, problems that arise tend to lack a clear structure and an obvious line of attack. As such, they are akin to many a problem in graph theory: their solution is likely to require ingenuity and original thought. Thus the expertise gained in solving the exercises in this book is likely to pay dividends not only in graph theory and other branches of mathematics, but also in other scientific disciplines.

"As long as a branch of science offers an abundance of problems, so long is it alive", said David Hilbert in his address to the Congress in Paris in 1900. Judged by this criterion, graph theory could hardly be more alive.

B. B.
Memphis
March 15, 1998

Preface

Graph theory is a young but rapidly maturing subject. Even during the quarter of a century that I lectured on it in Cambridge, it changed considerably, and I have found that there is a clear need for a text which introduces the reader not only to the well-established results, but to many of the newer developments as well. It is hoped that this volume will go some way towards satisfying that need.

There is too much here for a single course. However, there are many ways of using the book for a single-semester course: after a little preparation any chapter can be included in the material to be covered. Although strictly speaking there are almost no mathematical prerequisites, the subject matter and the pace of the book demand mathematical maturity from the student.

Each of the ten chapters consists of about five sections, together with a selection of exercises, and some bibliographical notes. In the opening sections of a chapter the material is introduced gently: much of the time results are rather simple, and the proofs are presented in detail. The later sections are more specialized and proceed at a brisker pace: the theorems tend to be deeper and their proofs, which are not always simple, are given rapidly. These sections are for the reader whose interest in the topic has been excited.

We do not attempt to give an exhaustive list of theorems, but hope to show how the results come together to form a cohesive theory. In order to preserve the freshness and elegance of the material, the presentation is not over-pedantic: occasionally the reader is expected to formalize some details of the argument. Throughout the book the reader will discover connections with various other branches of mathematics, like optimization theory, group theory, matrix algebra, probability theory, logic, and knot theory. Although the reader is not expected to have intimate knowledge of these fields, a modest acquaintance with them would enhance the enjoyment of this book.

The bibliographical notes are far from exhaustive: we are careful in our attributions of the major results, but beyond that we do little more than give suggestions for further readings.

A vital feature of the book is that it contains hundreds of exercises. Some are very simple, and test only the understanding of the concepts, but many go way beyond that, demanding mathematical ingenuity. We have shunned routine drills: even in the simplest questions the overriding criterion for inclusion was beauty. An attempt has been made to grade the exercises: those marked by $^-$ signs are five-finger exercises, while the ones with $^+$ signs need some inventiveness. Solving an exercise marked with $^{++}$ should give the reader a sense of accomplishment. Needless to say, this grading is subjective: a reader who has some problems with a standard exercise may well find a $^+$ exercise easy.

The conventions adopted in the book are standard. Thus, Theorem 8 of Chapter IV is referred to as Theorem 8 within the chapter, and as Theorem IV.8 elsewhere. Also, the symbol, $\square$, denotes the end of a proof; we also use it to indicate the absence of one.

The quality of the book would not have been the same without the valuable contributions of a host of people, and I thank them all sincerely. The hundreds of talented and enthusiastic Cambridge students I have lectured and supervised in graph theory; my past research students and others who taught the subject and provided useful feedback; my son, Márk, who typed and retyped the manuscript a number of times. Several of my past research students were also generous enough to give the early manuscript a critical reading: I am particularly grateful to Graham Brightwell, Yoshiharu Kohayakawa, Imre Leader, Oliver Riordan, Amites Sarkar, Alexander Scott and Andrew Thomason for their astute comments and perceptive suggestions. The deficiencies that remain are entirely my fault.

Finally, I would like to thank Springer-Verlag and especially Ina Lindemann, Anne Fossella and Anthony Guardiola for their care and efficiency in producing this book.

B. B.
Memphis
March 15, 1998

For help with preparation of the third printing, I would like to thank Richard Arratia, Peter Magyar, and Oliver Riordan. I am especially grateful to Don Knuth for sending me lists of misprints. For the many that undoubtedly remain, I apologize. Please refer to the website for this book, where I will maintain a list of further misprints that come to my attention; I'd be grateful for any assistance in making this list as complete as possible. The url for this book is http://www.msci.memphis.edu/faculty/bollobasb.html

B. B.
Memphis
April 16, 2002

Neque ingenium sine disciplina,
aut disciplina sine ingenio
perfectum artificem potest efficere.

Vitruvius

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I

Fundamentals

The basic concepts of graph theory are extraordinarily simple and can be used to express problems from many different subjects. The purpose of this chapter is to familiarize the reader with the terminology and notation that we shall use in the book. In order to give the reader practice with the definitions, we prove some simple results as soon as possible. With the exception of those in Section 5, all the proofs in this chapter are straightforward and could have safely been left to the reader. Indeed, the adventurous reader may wish to find his own proofs before reading those we have given, to check that he is on the right track.

The reader is not expected to have complete mastery of this chapter before sampling the rest of the book; indeed, he is encouraged to skip ahead, since most of the terminology is self-explanatory. We should add at this stage that the terminology of graph theory is still not standard, though the one used in this book is well accepted.

I.1 Definitions

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of V. Unless it is explicitly stated otherwise, we consider only finite graphs, that is, V and E are always finite. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set. An edge $\{x, y\}$ is said to *join* the vertices x and y and is denoted by xy. Thus xy and yx mean exactly the same edge; the vertices x and y are the *endvertices* of this edge. If $xy \in E(G)$, then x and y are