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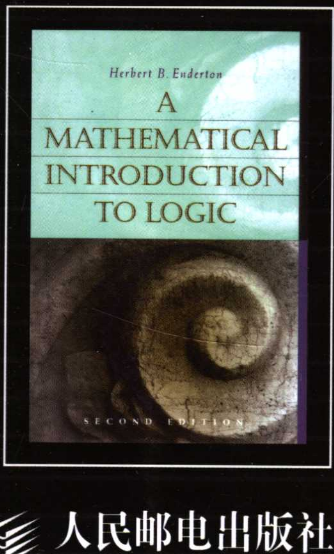
A Mathematical Introduction to Logic
Second Edition

数理逻辑

（英文版·第2版）

[美] Herbert B. Enderton 著



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图书在版编目（CIP）数据

数理逻辑：第2版／（美）恩德滕著．—北京：人民邮电出版社，2006.1
（图灵原版计算机科学系列）
ISBN 7-115-14145-2

Ⅰ．数…　Ⅱ．恩…　Ⅲ．数理逻辑－教材－英文　Ⅳ．TP0141

中国版本图书馆CIP数据核字（2005）第131661号

内 容 提 要

本书是数理逻辑方面的经典教材。书中涵盖了命题逻辑、一阶逻辑、不可判定性以及二阶逻辑等方面的内容，并且包含了与计算机科学有关的主题，如有限模型。本书特点是：内容可读性强；组织结构更灵活，授课教师可根据教学需要节选本书的内容；反映了近几年来理论计算机科学对逻辑学产生的影响；包含较多的示例和说明。本书适合作为计算机及相关专业本科生和研究生数理逻辑课程的教材。

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数理逻辑（英文版·第2版）

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◆ 人民邮电出版社出版发行　　北京市崇文区夕照寺街14号
　邮编　100061　　电子函件　315@ptpress.com.cn
　网址　http://www.ptpress.com.cn
　北京顺义振华印刷厂印刷
　新华书店总店北京发行所经销

◆ 开本：800×1000　1/16
　印张：20.5
　字数：440千字　　　　2006年1月第1版
　印数：1－3 000册　　　2006年1月北京第1次印刷

著作权合同登记号　图字：01-2005-5231号

ISBN 7-115-14145-2/TP・5059

定价：39.00元

读者服务热线：(010)88593802　印装质量热线：(010)67129223

Preface

This book, like the first edition, presents the basic concepts and results of logic: the topics are proofs, truth, and computability. As before, the presentation is directed toward the reader with some mathematical background and interests. In this revised edition, in addition to numerous "local" changes, there are three "global" ways in which the presentation has been changed:

First, I have attempted to make the material more *accessible* to the typical undergraduate student. In the main development, I have tried not to take for granted information or insights that might be unavailable to a junior-level mathematics student.

Second, for the instructor who wants to fit the book to his or her course, the organization has been made more *flexible*. Footnotes at the beginning of many of the sections indicate optional paths the instructor — or the independent reader — might choose to take.

Third, theoretical *computer science* has influenced logic in recent years, and some of that influence is reflected in this edition. Issues of computability are taken more seriously. Some material on finite models has been incorporated into the text.

The book is intended to serve as a textbook for an introductory mathematics course in logic at the junior-senior level. The objectives are to present the important concepts and theorems of logic and to explain their significance and their relationship to the reader's other mathematical work.

As a text, the book can be used in courses anywhere from a quarter to a year in length. In one quarter, I generally reach the material on models of first-order theories (Section 2.6). The extra time afforded by a semester would permit some glimpse of undecidability, as in Section 3.0. In a second term, the material of Chapter 3 (on undecidability) can be more adequately covered.

The book is intended for the reader who has not studied logic previously, but who has some experience in mathematical reasoning. There are no specific prerequisites aside from a willingness to function at a certain level of abstraction and rigor. There is the inevitable use of basic set theory. Chapter 0 gives a concise summary of the set theory used. One should not begin the book by studying this chapter; it is instead intended for reference if and when the need arises. The instructor can adjust the amount of set theory employed; for example it is possible to avoid cardinal numbers completely (at the cost of losing some theorems). The book contains some examples drawn from abstract algebra. But they are just examples, and are not essential to the exposition. The later chapters (Chapter 3 and 4) tend to be more demanding of the reader than are the earlier chapters.

Induction and recursion are given a more extensive discussion (in Section 1.4) than has been customary. I prefer to give an informal account of these subjects in lectures and have a precise version in the book rather than to have the situation reversed.

Exercises are given at the end of nearly all the sections. If the exercise bears a boldface numeral, then the results of that exercise are used in the exposition in the text. Unusually challenging exercises are marked with an asterisk.

I cheerfully acknowledge my debt to my teachers, a category in which I include also those who have been my colleagues or students. I would be pleased to receive comments and corrections from the users of this book. The Web site for the book can be found at http://www.math.ucla.edu/~hbe/amil.

Introduction

Symbolic logic is a mathematical model of deductive thought. Or at least that was true originally; as with other branches of mathematics it has grown beyond the circumstances of its birth. Symbolic logic is a model in much the same way that modern probability theory is a model for situations involving chance and uncertainty.

How are models constructed? You begin with a real-life object, for example an airplane. Then you select some features of this original object to be represented in the model, for example its shape, and others to be ignored, for example its size. And then you build an object that is like the original in some ways (which you call essential) and unlike it in others (which you call irrelevant). Whether or not the resulting model meets its intended purpose will depend largely on the selection of the properties of the original object to be represented in the model.

Logic is more abstract than airplanes. The real-life objects are certain "logically correct" deductions. For example,

> All men are mortal.
> Socrates is a man.
> Therefore, Socrates is mortal.

The validity of inferring the third sentence (the conclusion) from the first two (the assumptions) does not depend on special idiosyncrasies of Socrates. The inference is justified by the form of the sentences rather than by empirical facts about mortality. It is not really important here what "mortal" means; it does matter what "all" means.

> Borogoves are mimsy whenever it is brillig.
> It is now brillig, and this thing is a borogove.
> Hence this thing is mimsy.

Again we can recognize that the third sentence follows from the first two, even without the slightest idea of what a mimsy borogove might look like.

Logically correct deductions are of more interest than the above frivolous examples might suggest. In fact, axiomatic mathematics consists of many such deductions laid end to end. These deductions made by the working mathematician constitute real-life originals whose features are to be mirrored in our model.

The logical correctness of these deductions is due to their form but is independent of their content. This criterion is vague, but it is just this sort of vagueness that prompts us to turn to mathematical models. A major goal will be to give, within a model, a precise version of this criterion. The questions (about our model) we will initially be most concerned with are these:

1. What does it mean for one sentence to "follow logically" from certain others?
2. If a sentence does follow logically from certain others, what methods of *proof* might be necessary to establish this fact?
3. Is there a gap between what we can *prove* in an axiomatic system (say for the natural numbers) and what is *true* about the natural numbers?
4. What is the connection between logic and computability?

Actually we will present two models. The first (sentential logic) will be very simple and will be woefully inadequate for interesting deductions. Its inadequacy stems from the fact that it preserves only some crude properties of real-life deductions. The second model (first-order logic) is admirably suited to deductions encountered in mathematics. When a working mathematician asserts that a particular sentence follows from the axioms of set theory, he or she means that this deduction can be translated to one in our model.

This emphasis on mathematics has guided the choice of topics to include. This book does not venture into many-valued logic, modal logic, or intuitionistic logic, which represent different selections of properties of real-life deductions.

Thus far we have avoided giving away much information about what our model, first-order logic, is like. As brief hints, we now give some examples of the expressiveness of its formal language. First, take the English sentence that asserts the set-theoretic principle of extensionality, "If the same things are members of a first object as are members of a second object, then those objects are the same." This can be translated into our first-order language as

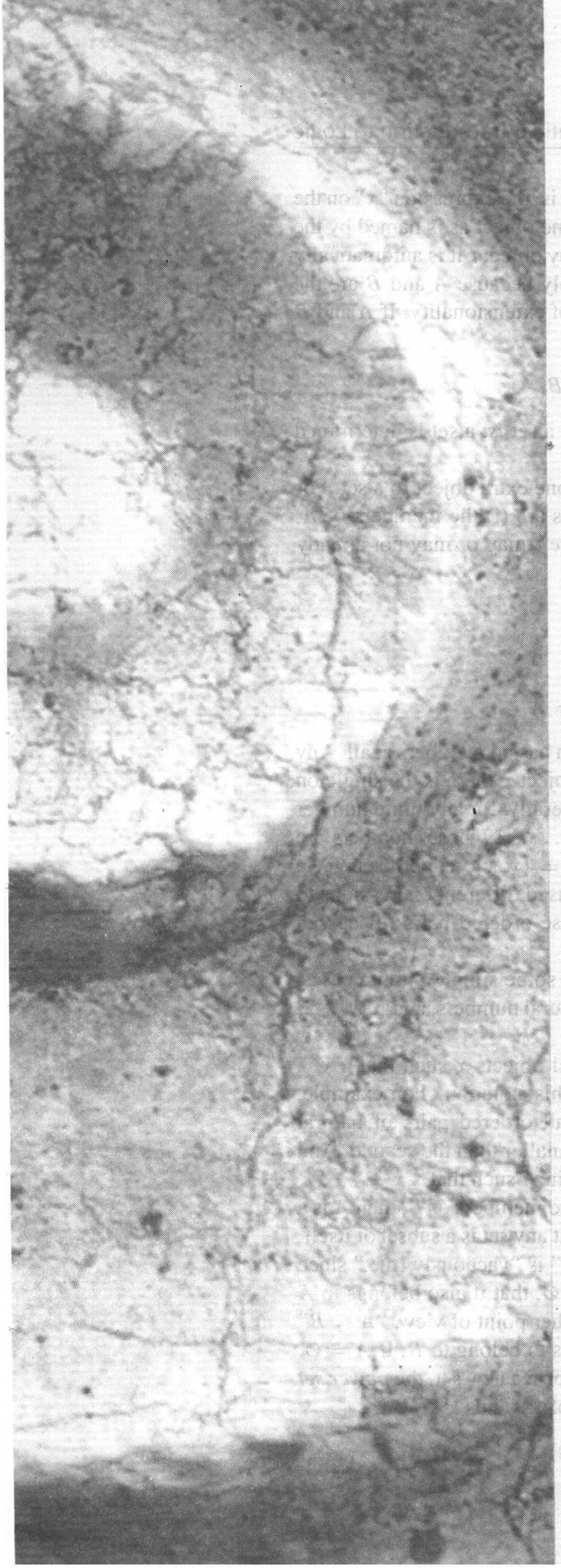
$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y).$$

As a second example, we can translate the sentence familiar to calculus students, "For every positive number ε there is a positive number δ such that for any x whose distance from a is less than δ, the distance between $f(x)$ and b is less than ε" as

$$\forall \varepsilon (\varepsilon > 0 \to \exists \delta (\delta > 0 \land \forall x (dxa < \delta \to dfxb < \varepsilon))).$$

We have given some hints as to what we intend to do in this book. We should also correct some possible misimpressions by saying what we are not going to do. This book does not propose to teach the reader how to think. The word "logic" is sometimes used to refer to remedial thinking, but not by us. The reader already knows how to think. Here are some intriguing concepts to think about.

Contents



Chapter ZERO

Useful Facts about Sets

We assume that the reader already has some familiarity with normal everyday set-theoretic apparatus. Nonetheless, we give here a brief summary of facts from set theory we will need; this will at least serve to establish the notation. It is suggested that the reader, instead of poring over this chapter at the outset, simply refer to it if and when issues of a set-theoretic nature arise in later chapters. The author's favorite book on set theory is of course his *Elements of Set Theory* (see the list of references at the end of this book).

First a word about jargon. Throughout the book we will utilize an assortment of standard mathematical abbreviations. We use "$\dashv$" to signify the end of a proof. A sentence "If ..., then ..." will sometimes be abbreviated "... $\Rightarrow$ ...." We also have "$\Leftarrow$" for the converse implication (for the peculiar way the word "implication" is used in mathematics). For "if and only if" we use the shorter "iff" (this has become part of the mathematical language) and the symbol "$\Leftrightarrow$." For the word "therefore" we have the "$\therefore$" abbreviation.

The notational device that extracts "$x \neq y$" as the denial of "$x = y$" and "$x \notin y$" as the denial of "$x \in y$" will be extended to other cases. For example, in Section 1.2 we define "$\Sigma \models \tau$"; then "$\Sigma \not\models \tau$" is its denial.

Now then, a *set* is a collection of things, called its members or elements. As usual, we write "$t \in A$" to say that t is a member of A, and "$t \notin A$" to say that t is not a member of A. We write "$x = y$" to mean that x and y are the same object. That is, the expression "x" on the left of the equals sign is a name for the same object as is named by the other expression "y." If $A = B$, then for any object t it is automatically true that $t \in A$ iff $t \in B$. This holds simply because A and B are the same thing. The converse is the principle of extensionality: If A and B are sets such that for every object t,

$$t \in A \quad \text{iff} \quad t \in B,$$

then $A = B$. This reflects the idea of what a set *is*; a set is determined just by its members.

A useful operation is that of adjoining one extra object to a set. For a set A, let $A; t$ be the set whose members are (i) the members of A, plus (ii) the (possibly new) member t. Here t may or may not already belong to A, and we have

$$A; t = A \cup \{t\}$$

using notation defined later, and

$$t \in A \quad \text{iff} \quad A; t = A.$$

One special set is the empty set $\varnothing$, which has no members at all. Any other set is said to be *nonempty*. For any object x there is the singleton set $\{x\}$ whose only member is x. More generally, for any finite number $x_1, \ldots, x_n$ of objects there is the set $\{x_1, \ldots, x_n\}$ whose members are exactly those objects. Observe that $\{x, y\} = \{y, x\}$, as both sets have exactly the same members. We have only used different expressions to denote the set. If order matters, we can use ordered pairs (discussed later).

This notation will be stretched to cover some simple infinite cases. For example, $\{0, 1, 2, \ldots\}$ is the set $\mathbb{N}$ of natural numbers, and $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set $\mathbb{Z}$ of all integers.

We write "$\{x \mid \_x\_\}$" for the set of all objects x such that $\_x\_$. We will take considerable liberty with this notation. For example, $\{\langle m, n\rangle \mid m < n \text{ in } \mathbb{N}\}$ is the set of all ordered pairs of natural numbers for which the first component is smaller than the second. And $\{x \in A \mid \_x\_\}$ is the set of all elements x in A such that $\_x\_$.

If A is a set all of whose members are also members of B, then A is a subset of B, abbreviated "$A \subseteq B$." Note that any set is a subset of itself. Also, $\varnothing$ is a subset of every set. ("$\varnothing \subseteq A$" is "vacuously true," since the task of verifying, for every member of $\varnothing$, that it also belongs to A requires doing nothing at all. Or from another point of view, "$A \subseteq B$" can be false only if some member of A fails to belong to B. If $A = \varnothing$, this is impossible.) From the set A we can form a new set, the *power set* $\mathcal{P}A$ of A, whose members are the subsets of A. Thus

$$\mathcal{P}A = \{x \mid x \subseteq A\}.$$

For example,

$$\mathcal{P}\varnothing = \{\varnothing\},$$
$$\mathcal{P}\{\varnothing\} = \{\varnothing, \{\varnothing\}\}.$$

The *union* of A and B, $A \cup B$, is the set of all things that are members of A or B (or both). For example, $A; t = A \cup \{t\}$. Similarly, the *intersection* of A and B, $A \cap B$, is the set of all things that are members of both A and B. Sets A and B are *disjoint* iff their intersection is empty (i.e., if they have no members in common). A collection of sets is *pairwise disjoint* iff any two members of the collection are disjoint.

More generally, consider a set A whose members are themselves sets. The union, $\bigcup A$, of A is the set obtained by dumping all the members of A into a single set:

$$\bigcup A = \{x \mid x \quad \text{belongs to some member of } A\}.$$

Similarly for nonempty A,

$$\bigcap A = \{x \mid x \quad \text{belongs to all members of } A\}.$$

For example, if

$$A = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\},$$

then

$$\bigcup A = \{0, 1, 5, 6\},$$

$$\bigcap A = \{1\}.$$

Two other examples are

$$A \cup B = \bigcup\{A, B\},$$

$$\bigcup \mathcal{P}A = A.$$

In cases where we have a set A_n for each natural number n, the union of all these sets, $\bigcup\{A_n \mid n \in \mathbb{N}\}$, is usually denoted "$\bigcup_{n\in\mathbb{N}} A_n$" or just "$\bigcup_n A_n$."

The ordered pair $\langle x, y\rangle$ of objects x and y must be defined in such a way that

$$\langle x, y\rangle = \langle u, v\rangle \qquad \text{iff} \qquad x = u \quad \text{and} \quad y = v.$$

Any definition that has this property will do; the standard one is

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

For ordered triples we define

$$\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle.$$

More generally we define n-tuples recursively by

$$\langle x_1, \ldots, x_{n+1} \rangle = \langle \langle x_1, \ldots, x_n \rangle, x_{n+1} \rangle$$

for $n > 1$. It is convenient to define also $\langle x \rangle = x$; the preceding equation then holds also for $n = 1$. S is a *finite sequence* (or *string*) of members of A iff for some positive integer n, we have $S = \langle x_1, \ldots, x_n \rangle$, where each $x_i \in A$. (Finite sequences are often defined to be certain finite functions, but the above definition is slightly more convenient for us.)

A *segment* of the finite sequence $S = \langle x_1, \ldots, x_n \rangle$ is a finite sequence

$$\langle x_k, x_{k+1}, \ldots, x_{m-1}, x_m \rangle, \qquad \text{where} \quad 1 \le k \le m \le n.$$

This segment is an *initial segment* iff $k = 1$ and it is *proper* iff it is different from S.

If $\langle x_1, \ldots, x_n \rangle = \langle y_1, \ldots, y_n \rangle$, then it is easy to see that $x_i = y_i$ for $1 \le i \le n$. (The proof uses induction on n and the basic property of ordered pairs.) But if $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle$, then it does not in general follow that $m = n$. After all, every ordered triple is also an ordered pair. But we claim that m and n can be unequal only if some x_i is itself a finite sequence of y_j's, or the other way around:

Lemma 0A Assume that $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m, \ldots, y_{m+k} \rangle$. Then $x_1 = \langle y_1, \ldots, y_{k+1} \rangle$.

Proof. We use induction on m. If $m = 1$, the conclusion is immediate. For the inductive step, assume that $\langle x_1, \ldots, x_m, x_{m+1} \rangle = \langle y_1, \ldots, y_{m+k}, y_{m+1+k} \rangle$. Then the first components of this ordered pair must be equal: $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_{m+k} \rangle$. Now apply the inductive hypothesis. ⊣

For example, suppose that A is a set such that no member of A is a finite sequence of other members. Then if $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle$ and each x_i and y_j is in A, then by the above lemma $m = n$. Whereupon we have $x_i = y_i$ as well.

From sets A and B we can form their *Cartesian product*, the set $A \times B$ of all pairs $\langle x, y \rangle$ for which $x \in A$ and $y \in B$. A^n is the set of all n-tuples of members of A. For example, $A^3 = (A \times A) \times A$.

A *relation* R is a set of ordered pairs. For example, the ordering relation on the numbers 0–3 is captured by — and in fact *is* — the set of ordered pairs

$$\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}.$$

The *domain* of R (written dom R) is the set of all objects x such that $\langle x, y \rangle \in R$ for some y. The *range* of R (written ran R) is the set of all objects y such that $\langle x, y \rangle \in R$ for some x. The union of dom R and ran R is the *field* of R, fld R.

An *n-ary relation* on A is a subset of A^n. If $n > 1$, it is a relation. But a 1-ary (unary) relation on A is simply a subset of A. A particularly simple binary relation on A is the equality relation $\{\langle x, x\rangle \mid x \in A\}$ on A. For an n-ary relation R on A and subset B of A, the *restriction* of R to B is the intersection $R \cap B^n$. For example, the relation displayed above is the restriction to the set $B = \{0, 1, 2, 3\}$ of the ordering relation on $\mathbb{N}$.

A *function* is a relation F with the property of being *single-valued*: For each x in dom F there is only one y such that $\langle x, y\rangle \in F$. As usual, this unique y is said to be the value $F(x)$ that F assumes at x. (This notation goes back to Euler. It is a pity he did not choose $(x)F$ instead; that would have been helpful for the *composition* of functions: $f \circ g$ is the function whose value at x is $f(g(x))$, obtained by applying first g and then f.)

We say that F *maps* A into B and write

$$F : A \to B$$

to mean that F is a function, dom $F = A$, and ran $F \subseteq B$. If in addition ran $F = B$, then F maps A *onto* B. F is *one-to-one* iff for each y in ran F there is only one x such that $\langle x, y\rangle \in F$. If the pair $\langle x, y\rangle$ is in dom F, then we let $F(x, y) = F(\langle x, y\rangle)$. This notation is extended to n-tuples; $F(x_1, \ldots, x_n) = F(\langle x_1, \ldots, x_n\rangle)$.

An *n-ary operation* on A is a function mapping A^n into A. For example, addition is a binary operation on $\mathbb{N}$, whereas the successor operation S (where $S(n) = n + 1$) is a unary operation on $\mathbb{N}$. If f is an n-ary operation on A, then the *restriction* of f to a subset B of A is the function g with domain B^n which agrees with f at each point of B^n. Thus,

$$g = f \cap (B^n \times A).$$

This g will be an n-ary operation on B iff B is *closed* under f, in the sense that $f(b_1, \ldots, b_n) \in B$ whenever each b_i is in B. In this case, $g = f \cap B^{n+1}$, in agreement with our definition of the restriction of a relation. For example, the addition operation on $\mathbb{N}$, which contains such triples as $\langle\langle 3, 2\rangle, 5\rangle$, is the restriction to $\mathbb{N}$ of the addition operation on $\mathbb{R}$, which contains many more triples.

A particularly simple unary operation on A is the *identity* function Id on A, given by the equation

$$Id(x) = x \qquad \text{for} \quad x \in A.$$

Thus $Id = \{\langle x, x\rangle \mid x \in A\}$.

For a relation R, we define the following:

R is *reflexive* on A iff $\langle x, x\rangle \in R$ for every x in A.

R is *symmetric* iff whenever $\langle x, y\rangle \in R$, then also $\langle y, x\rangle \in R$.

R is *transitive* iff whenever both $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ (if this ever happens), then also $\langle x, z\rangle \in R$.

R satisfies *trichotomy* on A iff for every x and y in A, exactly one of the three possibilities, $\langle x, y\rangle \in R$, $x = y$, or $\langle y, x\rangle \in R$, holds.

R is an *equivalence relation* on A iff R is a binary relation on A that is reflexive on A, symmetric, and transitive.

R is an *ordering relation* on A iff R is transitive and satisfies trichotomy on A.

For an equivalence relation R on A we define, for $x \in A$, the *equivalence class* $[x]$ of x to be $\{y \mid \langle x, y\rangle \in R\}$. The equivalence classes then *partition* A. That is, the equivalence classes are subsets of A such that each member of A belongs to exactly one equivalence class. For x and y in A,

$$[x] = [y] \quad \text{iff} \quad \langle x, y\rangle \in R.$$

The set $\mathbb{N}$ of natural numbers is the set $\{0, 1, 2, \ldots\}$. (Natural numbers can also be defined set-theoretically, a point that arises briefly in Section 3.7.) A set A is *finite* iff there is some one-to-one function f mapping (for some natural number n) the set A onto $\{0, 1, \ldots, n-1\}$. (We can think of f as "counting" the members of A.)

A set A is *countable* iff there is some function mapping A one-to-one into $\mathbb{N}$. For example, any finite set is obviously countable. Now consider an infinite countable set A. Then from the given function f mapping A one-to-one into $\mathbb{N}$, we can extract a function f' mapping A one-to-one *onto* $\mathbb{N}$. For some $a_0 \in A$, $f(a_0)$ is the least member of ran f, let $f'(a_0) = 0$. In general there is a unique $a_n \in A$ such that $f(a_n)$ is the $(n+1)$st member of ran f; let $f'(a_n) = n$. Note that $A = \{a_0, a_1, \ldots\}$. (We can also think of f' as "counting" the members of A, only now the counting process is infinite.)

Theorem 0B Let A be a countable set. Then the set of all finite sequences of members of A is also countable.

Proof. The set S of all such finite sequences can be characterized by the equation

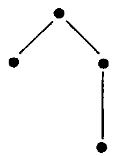
$$S = \bigcup_{n \in \mathbb{N}} A^{n+1}.$$

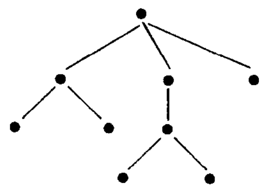
Since A is countable, we have a function f mapping A one-to-one into $\mathbb{N}$.

The basic idea is to map S one-to-one into $\mathbb{N}$ by assigning to $\langle a_0, a_1, \ldots, a_m\rangle$ the number $2^{f(a_0)+1} 3^{f(a_1)+1} \cdot \ldots \cdot p_m^{f(a_m)+1}$, where p_m is the $(m+1)$st prime. This suffers from the defect that this assignment might not be well-defined. For conceivably there could be $\langle a_0, a_1, \ldots, a_m\rangle = \langle b_0, b_1, \ldots, b_n\rangle$, with a_i and b_j in A but with $m \neq n$. But this is not serious; just assign to each member of S the *smallest* number obtainable in the above fashion. This gives us a well-defined map; it is easy to see that it is one-to-one. ⊣

At times we will speak of *trees*, which can be useful in providing intuitive pictures of some situations. But our comments on trees will always be informal; the theorems and proofs will not rely on trees. Accordingly, our discussion here of trees will be informal.

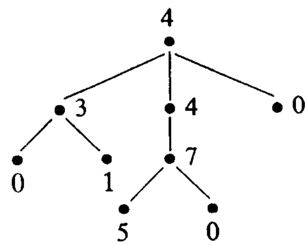
For each tree there is an underlying finite partial ordering. We can draw a picture of this partial ordering R; if $\langle a, b\rangle \in R$, then we put a lower than b and connect the points by a line. Pictures of two typical tree orderings are shown.





(In mathematics, trees grow downward, not upward.) There is always a highest point in the picture (the *root*). Furthermore, while branching is permitted below some vertex, the points *above* any given vertex must lie along a line.

In addition to this underlying finite partial ordering, a tree also has a labeling function whose domain is the set of vertices. For example, one tree, in which the labels are natural numbers, is shown.



At a few points in the book we will use the axiom of choice. But usually these uses can be eliminated if the theorems in question are restricted to countable languages. Of the many equivalent statements of the axiom of choice, Zorn's lemma is especially useful.

Say that a collection C of sets is a *chain* iff for any elements x and y of C, either $x \subseteq y$ or $y \subseteq x$.

Zorn's Lemma Let A be a set such that for any chain $C \subseteq A$, the set $\bigcup C$ is in A. Then there is some element $m \in A$ which is maximal in the sense that it is not a subset of any other element of A.

Cardinal Numbers

All infinite sets are big, but some are bigger than others. (For example, the set of real numbers is bigger than the set of integers.) Cardinal numbers provide a convenient, although not indispensable, way of talking about the size of sets.

It is natural to say that two sets A and B have the same size iff there is a function that maps A one-to-one onto B. If A and B are finite, then this concept is equivalent to the usual one: If you count the members of A and the members of B, then you get the same number both times. But it is applicable even to infinite sets A and B, where counting is difficult.

Formally, then, say that A and B are *equinumerous* (written $A \sim B$) iff there is a one-to-one function mapping A onto B. For example, the set $\mathbb{N}$ of natural numbers and the set $\mathbb{Z}$ of integers are equinumerous. It is easy to see that equinumerosity is reflexive, symmetric, and transitive.

For finite sets we can use natural numbers as measures of size. The same natural number would be assigned to two finite sets (as measures of their size) iff the sets were equinumerous. Cardinal numbers are introduced to enable us to generalize this situation to infinite sets.

To each set A we can assign a certain object, the *cardinal number* (or *cardinality*) of A (written card A), in such a way that two sets are assigned the same cardinality iff they are equinumerous:

$$\operatorname{card} A = \operatorname{card} B \quad \text{iff} \quad A \sim B. \tag{K}$$

There are several ways of accomplishing this; the standard one these days takes card A to be the least ordinal equinumerous with A. (The success of this definition relies on the axiom of choice.) We will not discuss ordinals here, since for our purposes it matters very little what card A actually is, any more than it matters what the number 2 actually is. What matters most is that (K) holds. It is helpful, however, if for a finite set A, card A is the natural number telling how many elements A has. Something is a *cardinal number*, or simply a *cardinal*, iff it is card A for some set A.

(Georg Cantor, who first introduced the concept of cardinal number, characterized in 1895 the cardinal number of a set M as "the general concept which, with the help of our active intelligence, comes from the set M upon abstraction from the nature of its various elements and from the order of their being given.")

Say that A is *dominated* by B (written $A \preceq B$) iff A is equinumerous with a subset of B. In other words, $A \preceq B$ iff there is a one-to-one function mapping A into B. The companion concept for cardinals is

$$\operatorname{card} A \leq \operatorname{card} B \quad \text{iff} \quad A \preceq B.$$

(It is easy to see that $\leq$ is well defined; that is, whether or not $\kappa \leq \lambda$ depends only on the cardinals κ and λ themselves, and not the choice of