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An Introduction to Nonlinear Finite Element Analysis

with applications to heat transfer, fluid mechanics, and solid mechanics

SECOND EDITION

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Preface to the Second Edition

The development of realistic mathematical models that govern the response of systems or processes is intimately connected to the ability to translate them into meaningful discrete models that enable us to systematically evaluate various parameters of the systems and processes. Mathematical model development and numerical simulations are aids to designers, who are seeking to maximize the reliability of products and minimize the cost of production, distribution, and repairs. Mathematical models are developed using laws of physics and assumptions concerning a system's behavior. The most important step in arriving at a design that is both reliably functional and cost-effective is the construction of a suitable mathematical model of the system behavior and its translation into a powerful numerical simulation tool. While a select number of courses on continuum mechanics, material science, and dynamical systems, among others, provides engineers with the background to formulate a suitable mathematical model, courses on numerical methods prepare engineers and scientists to evaluate mathematical models and test material models in the context of the functionality and design constraints placed on the system. In cases where physical experiments are prohibitively expensive, numerical simulations are the only alternative, especially when the phenomena is governed by nonlinear differential equations, in evaluating various design options. It is in this context a course on nonlinear finite element analysis proves to be very useful.

Most books on nonlinear finite element analysis tend to be abstract in the presentation of details of the finite element formulations, derivation of element equations, and their solution by iterative methods. Such books serve as reference books but not as textbooks. The present textbook is unique (i.e. there is no parallel to this book in its class) since it actually helps the readers with details of finite element model development and implementation. In particular, it provides illustrative examples and problem sets that enable readers to test their understanding of the subject matter and utilize the tools developed in the formulation and finite element analysis of engineering problems.

The second edition of *An Introduction to Nonlinear Finite Element Analysis* has the same objective as the first edition, namely, to facilitate an easy and thorough understanding of the details that are involved in the theoretical formulation, finite element model development, and solutions of nonlinear problems. *The book offers easy-to-understand treatment of the subject of nonlinear finite element analysis, which includes element development from mathematical models and numerical evaluation of the underlying physics.* The new edition is extensively reorganized and contains substantially large amount of new material. In particular, Chapter 1 in the second edition contains a section on applied functional analysis; Chapter 2 on nonlinear continuum mechanics is entirely new; Chapters 3 through 8 in the new edition correspond to Chapter 2 through 8 of the first edition but with additional explanations, examples, and exercise problems (material on time dependent problems from Chapter 8 of the first edition is absorbed into Chapters 6 through 10 of the new edition); Chapter 9 is extensively revised and it contains up to date developments in the large deformation analysis of isotropic, composite, and functionally graded shells; Chapter 10 of the first edition on material nonlinearity and coupled problems is reorganized in the second edition by moving the material on solid mechanics to Chapter 12 in the new edition, and material on coupled problems to Chapter 10 on weak-form Galerkin finite element models of viscous incompressible fluids; finally, Chapter 11 in the second edition is entirely new and devoted to least-squares finite element models of viscous incompressible fluids. Chapter 12 of the second edition (available only online) contains material on one-dimensional formulations of nonlinear elasticity, plasticity, and viscoelasticity. In general, all of the chapters of the second edition contain additional explanations, detailed example problems, and additional exercise problems. Although all of the programming segments are in Fortran, the logic used in these Fortran programs is transparent and can be used in Matlab or C^{++} versions of the same. Thus the new edition more than replaces the first edition, and it is hoped that it is acquired by the library of every institution of higher learning as well as serious finite element analysts.

The book may be used as a textbook for an advanced course (after a first course) on the finite element method or the first course on nonlinear finite element analysis. A solutions manual has also been prepared for the book. The solution manual is available from the publisher only to instructors who adopt the book as a textbook for a course.

Since the publication of the first edition, many users of the book communicated their comments and compliments as well as errors they found, for which the author thanks them. All of the errors known to the author have been corrected in the current edition. The author is grateful to the following professional colleagues for their friendship, encouragement, and constructive comments on the book:

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Preface to the First Edition

The objective of this book is to present the theory and computer implementation of the finite element method as applied to simple nonlinear problems of heat transfer and similar field problems, fluid mechanics, and solid mechanics. Both geometric as well as material nonlinearities are considered, and static and transient (i.e. time-dependent) responses are studied. The guiding principle in writing the book was to make the presentation suitable for (a) adoption as a text book for a first course on nonlinear finite element analysis (or for a second course following an introductory course on the finite element method) and (b) for use by engineers and scientists from various disciplines for self-study and practice.

There exist a number of books on nonlinear finite elements. Most of these books contain a good coverage of the topics of structural mechanics, and few address topics of fluid dynamics and heat transfer. While these books serve as good references to engineers or scientists who are already familiar with the subject but wish to learn advanced topics or latest developments, they are not suitable as textbooks for a first course or for self study on nonlinear finite element analysis.

The motivation and encouragement that led to the writing of the present book have come from the users of the author's book, *An Introduction to the Finite Element Method* (McGraw-Hill, 1984; Second Edition, 1993; third edition scheduled for 2004), who have found the approach presented there to be most suitable for any one – irrespective of their scientific background – interested in learning the method, and also from the fact that there does not exist a book that is suitable as a textbook for a first course on nonlinear finite element analysis. The author has taught a course on nonlinear finite element analysis many times during the last twenty years, and the present book is an outcome of the lecture notes developed during this period. The same approach as that used in the aforementioned book, namely, the *differential equation approach*, is adopted in the present book to introduce the theory, formulation, and computer implementation of the finite element method as applied to nonlinear problems of science and engineering.

Beginning with a model (i.e. typical) second-order, nonlinear differential equation in one dimension, the book takes the reader through increasingly complex problems of nonlinear beam bending, nonlinear field problems in two dimensions, nonlinear plate bending, nonlinear formulations of solid continua, flows of viscous incompressible fluids in two dimensions (i.e. Navier–Stokes equations), time-approximation schemes, continuum formulations of shells, and material nonlinear problems of solid mechanics.

As stated earlier, the book is suitable as a textbook for a first course on nonlinear finite elements in civil, aerospace, mechanical, and mechanics departments as well as in applied sciences. It can be used as a reference by engineers and scientists working in industry, government laboratories, and academia. Introductory courses on the finite element method, continuum mechanics, and numerical analysis should prove to be helpful.

The author has benefited in writing the book by the encouragement and support of many colleagues around the world who have used his book, *An Introduction to the Finite Element Method*, and students who have challenged him to explain and implement complicated concepts and formulations in simple ways. While it is not possible to name all of them, the author expresses his sincere appreciation. The author expresses his deep sense of gratitude to his teacher and mentor, Professor J. T. Oden (University of Texas at Austin), without whose advice and support it would not have been possible for the author to modestly contribute to the field of applied mechanics in general and theory and application of the finite element method in particular, through his teaching, research, and writings.

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About the Author

J. N. Reddy is a University Distinguished Professor, Regents Professor, and the Holder of Oscar S. Wyatt Endowed Chair in the Department of Mechanical Engineering at Texas A&M University. Prior to the current position, he was the Clifton C. Garvin Professor in the Department of Engineering Science and Mechanics at Virginia Tech, and Associate Professor in the School of Mechanical, Aerospace and Nuclear Engineering at the University of Oklahoma.

Dr. Reddy is internationally known for his contributions to theoretical and applied mechanics and computational mechanics. He is the author of more than 500 journal papers and 18 textbooks with multiple editions. Professor Reddy is the recipient of numerous awards including the *Worcester Reed Warner Medal*, the *Charles Russ Richards Memorial Award*, and the *Honorary Member* award from the American Society of Mechanical Engineers, the *Nathan M. Newmark Medal* and the *Raymond D. Mindlin Medal* from the American Society of Civil Engineers, the *Distinguished Research Award* and the *Excellence in the Field of Composites Award* from the American Society of Composites, the *Computational Solid Mechanics Award* from the U.S. Association of Computational Mechanics, the *Computational Mechanics Award* from the Japanese Society of Mechanical Engineers, the *IACM Award* from the International Association of Computational Mechanics, and the *Archie Higdon Distinguished Educator Award* from the American Society of Engineering Education. Dr. Reddy received honorary degrees (Honoris Causa) from the Technical University of Lisbon (Portugal), and from Odlar Yurdu University (Azerbaijan). He is a Fellow of the American Society of Mechanical Engineers, the American Institute of Aeronautics and Astronautics, the American Society of Civil Engineers , the American Academy of Mechanics, the American Society of Composites, the U.S. Association of Computational Mechanics, the International Association of Computational Mechanics, and the Aeronautical Society of India.

Professor Reddy is the Editor-in-Chief of *Mechanics of Advanced Materials and Structures* and *International Journal of Computational Methods in Engineering Science and Mechanics*, and co-Editor of *International Journal of Structural Stability and Dynamics*; he also serves on the editorial boards of more than two dozen other journals, including *International Journal for Numerical Methods in Engineering, Computer Methods in Applied Mechanics and Engineering*, and *International Journal of Non-Linear Mechanics*.

Dr. Reddy is a selective researcher in engineering around the world who is recognized by *ISI Highly Cited Researchers* with over 15,000 citations (without self-citations over 14,000) with h-index of over 58 as per Web of Science; as per Google Scholar, the current number of citations exceed 39,000, and the h-index is 77. A more complete resume with links to journal papers can be found at **http://www.tamu.edu/acml/**.

List of Symbols

The symbols that are used throughout the book for various quantities are defined in the following list but the list is not exhaustive. In some cases, the same symbol has different meaning in different parts of the book, as it would be clear in the context.

Arabic alphabetical symbols

$\mathbf{a}$	Acceleration vector, $\frac{D\mathbf{v}}{Dt}$
$B(\cdot,\cdot)$	Bilinear form
$\mathbf{B}$	Left Cauchy–Green deformation tensor (or Finger tensor), $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$; magnetic flux density vector
$\tilde{\mathbf{B}}$	Cauchy strain tensor, $\tilde{\mathbf{B}} = \mathbf{F}^{-\mathrm{T}}\cdot\mathbf{F}^{-1}$; $\tilde{\mathbf{B}}^{-1} = \mathbf{B}$
c	Specific heat, moisture concentration
c_v, c_p	Specific heat at constant volume and pressure
$\mathbf{c}$	Couple vector
$\mathbf{C}$	Right Cauchy–Green deformation tensor, $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$; fourth-order elasticity tensor [see Eq. (2.4.3)] with coefficients C_{ij} or C_{ijkl}
$\mathbf{d}$	Symmetric part of the velocity gradient tensor, $\mathbf{l} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}$, that is $\mathbf{d} = \frac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} + \boldsymbol{\nabla}\mathbf{v}\right]$; electric flux vector; mass diffusivity tensor
$\mathcal{D}$	Internal dissipation
$d\mathbf{a}$	Area element (vector) in spatial description
$d\mathbf{A}$	Area element (vector) in material description
$d\mathbf{x}$	Line element (vector) in current configuration
$d\mathbf{X}$	Line element (vector) in reference configuration
D/Dt, d/dt	Material time derivative
ds	Surface element in current configuration
dS	Surface element in reference configuration
e_c	Internal energy per unit mass
$\mathbf{e}$	Almansi strain tensor, $\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\cdot\mathbf{F}^{-1}\right)$
$\mathbf{e}_i$	A basis vector in the x_i-direction
$\mathbf{e}_{ijk}$	Components of alternating or permutation tensor, $\mathcal{E}$
$\hat{\mathbf{e}}$	A unit vector
$\hat{\mathbf{e}}_A$	A unit basis vector in the direction of vector $\mathbf{A}$
E, E_1, E_2	Young's modulus (modulus of elasticity)
$\mathbf{E}$	Green–Lagrange strain tensor, $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} - \mathbf{I}\right)$ with components E_{ij}
$\hat{\mathbf{E}}_i$	Unit base vector along the X_i material coordinate direction
$\mathbf{f}$	Body force vector
f_x, f_y, f_z	Body force components in the x, y, and z directions
f	Load per unit length of a bar

q_n	Heat flux normal to the boundary, $q_n = \boldsymbol{\nabla} \cdot \hat{\mathbf{n}}$
$\mathbf{q}_0$	Heat flux vector in the reference configuration
$\mathbf{q}_i$	Force components
$\mathbf{q}$	Heat flux vector in the current configuration
Q	First moment of area; volume rate of flow
Q_h	Heat input
r	Radial coordinate in the cylindrical polar system
r_h	Internal heat generation per unit mass in the current configuration
r_0	Internal heat generation per unit mass in the reference configuration
R	Radial coordinate in the spherical coordinate system; universal gas constant
$\mathbf{R}$	Position vector in the spherical coordinate system; proper orthogonal tensor; residual vector
$\mathbf{S}$	A second-order tensor; second Piola–Kirchhoff stress tensor
S_{ij}	Elastic compliance coefficients
t	Time
$\mathbf{t}$	Stress vector; traction vector
T	Torque; temperature
$\mathbf{T}$	Tangent coefficient matrix with coefficients T_{ij}
$\mathbf{u}$	Displacement vector
u_1, u_2, u_3	Displacements in the x_1, x_2, and x_3 directions
U	Internal (or strain) energy
$\mathbf{U}$	Right Cauchy stretch tensor
u, v, w	Displacement components in the x, y, and z directions
u_x, u_y, u_z	Displacements in the x, y, and z directions
v	Velocity, $v = \lvert\mathbf{v}\rvert$
V	Shear force in beam problems; potential energy due to loads
V_f	Scalar potential
$\mathbf{v}$	Velocity vector in spatial coordinates, $\mathbf{v} = \frac{D\mathbf{x}}{Dt}$
$\mathbf{V}$	Velocity vector in material coordinates; left Cauchy stretch tensor
v_x, v_y, v_z	Velocity components in the x, y, and z directions
$\mathbf{w}$	Vorticity vector, $\mathbf{w} = \frac{1}{2}\boldsymbol{\nabla} \times \mathbf{v}$
W_{net}	Net rate of power input
$\mathbf{W}$	Skew symmetric part of the velocity gradient tensor, $\mathbf{L} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}$; that is $\mathbf{W} = \frac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} - \boldsymbol{\nabla}\mathbf{v}\right]$,
$\mathbf{x}$	Spatial coordinates
$\mathbf{X}$	Material coordinates
x_1, x_2, x_3	Rectangular Cartesian coordinates
x, y, z	Rectangular Cartesian coordinates
Y	Relaxation modulus
z	Transverse coordinate in the beam problem; axial coordinate in the torsion problem

Greek and parenthetical symbols

α	Angle; coefficient of thermal expansion; a parameter in time-approximation schemes
α_{ij}	Thermal coefficients of expansion
β	Acceleration parameter for convergence
β_{ij}	Material coefficients, $\beta_{ij} = C_{ijk\ell}\,\alpha_{k\ell}$
γ	Parameter in the Newmark scheme; penalty parameter
$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	Shear strains in structural problems
Γ	Internal entropy production; boundary of Ω
δ	Variational operator used in Chapter 2; Dirac delta
δ_{ij}	Components of the unit tensor, $\mathbf{I}$ (Kronecker delta)
Δ	Change of (followed by another symbol)
ε	Infinitesimal strain tensor
$\tilde{\varepsilon}$	Symmetric part of the displacement gradient tensor, $(\nabla\mathbf{u})^{\mathrm{T}}$; that is $\tilde{\varepsilon} = \frac{1}{2}\left[(\nabla\mathbf{u})^{\mathrm{T}} + \nabla\mathbf{u}\right]$
ϵ	Total stored energy per unit mass; convergence tolerance
ε_{ij}	Rectangular components of the infinitesimal strain tensor
ζ	Natural coordinate
η	Entropy density per unit mass; dashpot constant; natural coordinate
η_0	Viscosity coefficient
θ	Angular coordinate in the cylindrical and spherical coordinate systems; angle; twist per unit length; absolute temperature
κ_0, κ	Reference and current configurations
λ	Extension ratio; Lamé constant; eigenvalue
μ	Lamé constant; viscosity; principal value of strain
ν	Poisson's ratio; ν_{ij} Poisson's ratios
ξ	Natural coordinate
Π	Total potential energy functional
ρ	Density in the current configuration; charge density
ρ_0	Density in the reference configuration
σ	Boltzman constant
$\tilde{\sigma}$	Mean stress
$\boldsymbol{\sigma}$	Cauchy stress tensor
τ	Shear stress; time
$\boldsymbol{\tau}$	Viscous stress tensor
$\boldsymbol{\chi}$	Deformation mapping
φ_i	Approximation functions; Hermite interpolation functions
ϕ	A typical variable; angular coordinate in the spherical coordinate system; electric potential; relaxation function

Contents