Differential Forms

with

Applications to the Physical Sciences

Harley Flanders

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HARLEY FLANDERS
Purdue University
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Foreword

After several friendly discussions of the pros and cons of tensors versus differential forms in the solution of engineering problems, I persuaded my colleague Dr. Flanders to prepare a number of lectures on differential forms. The result was an outstanding series of lectures which was presented to a group of interested faculty members within the several schools of Engineering at Purdue University.

It became obvious to those attending that the use of differential forms would give them another tool for the analysis and synthesis of engineering systems. There are certain problems, normally very difficult to solve by using tensors only, for which results are more quickly and directly obtained with differential forms.

The author was encouraged to formalize his notes to the extent necessary for publication, to enable others to study this important subject. The text is recommended highly because differential forms and related concepts which have evolved from modern mathematics are new and powerful analytical tools for use by the engineer and scientist.

GEORGE A. HAWKINS, Dean

Schools of Engineering and Mathematical Sciences
Purdue University
November 20, 1962

Preface

Last spring the author gave a series of lectures on exterior differential forms to a group of faculty members and graduate students from the Purdue Engineering Schools. The material that was covered in these lectures is presented here in an expanded version. The book is aimed primarily at engineers and physical scientists in the hope of making available to them new tools of very great power in modern mathematics. Although none of our applications goes very deep, it is hoped, nevertheless, that enough ground is covered in each case to indicate the usefulness of this machinery.

A word about the organization of the book is in order. The first chapter is introductory and sketches where we are going and why. Chapters II, III, and V include all of the theoretical material; a knowledge of this opens the door to the applications. Probably on first reading, one should aim more at developing some intuition for the subject and getting a firm idea of what the various different things which are defined look like, rather than at working out proofs in detail. Applications to questions in differential geometry (including many topics of considerable use in physical sciences) are mostly in Chapters IV, VI, VIII, and IX. Applications to various topics in ordinary and partial differential equations will be found in Chapter VII. Finally, applications to several topics in physics are in Sections 3.5, 4.6, 6.4, and Chapter X.

What is presupposed of the reader is first of all a certain amount of scientific maturity, the precise direction not being too important. While the book is not really advanced mathematics, it is not exactly ground floor mathematics either, and a reasonable knowledge of the calculus of functions of several real variables is necessary, as is a working knowledge of linear algebra through the ideas of linear combination, basis, dimension, linear transformation. Some exposure to a minimum amount of the ground rules of modern mathematics, sets, cartesian products, functions on sets, is helpful but not essential. This material is usually picked up by osmosis anyway, and the Glossary of Notation at the end of the book should be helpful. The reader should also know about the existence of solutions of ordinary differential equations. A passing familiarity with tensor methods is useful, but not essential.

If our audience consisted of mathematicians alone, it would be in order to use somewhat more care in our formulations of definitions and proofs of theorems and to discuss in considerably more depth numerous technical points we here pass over lightly. Our goal, however, is to develop an intuition and a working knowledge of the subject with as much dispatch as is possible. This perhaps could be done in less space except for our insistence on a degree of rigor matching that found in the better treatises on theoretical physics. This falls short of the extremely great precision which is customary in modern abstract mathematics and pretty much inherent in its nature. One who quite rightly is searching recent developments in mathematics for applicable material must find this precision a considerable barricade, overpedantic if not downright tedious—a very real factor in the great separation between modern mathematics and modern science. Making his craft available to science is not a light task for the mathematician and the extent to which this book makes a contribution therein must necessarily be its primary measure of success.

In spite of all this, we do not hesitate to recommend this material to graduate students in mathematics as an introduction to modern differential geometry; indeed, a well-trained advanced undergraduate should find the book quite accessible. Considering the degree to which present day mathematical training consists of one abstraction after another, some of the things in this book could be a bit of an eye-opener, even to a mathematics student who is well along. For example, one could envisage such a student meeting here a parabolic differential equation, or a matrix group, or a contact transformation for the very first time.

It is my pleasant duty to acknowledge the substantial help and encouragement I have always had from my teachers, colleagues, and students. In this respect a special vote of thanks is due George A. Hawkins, Dean of the Schools of Engineering and Mathematical Sciences of Purdue University. Finally, I wish to express my gratitude to Elizabeth Young, whose beautiful typing of the manuscript was a substantial contribution.

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I

Introduction

1.1. Exterior Differential Forms

The objects which we shall study are called *exterior differential forms.* These are the things which occur under integral signs. For example, a line integral

$$\int A\,dx + B\,dy + C\,dz$$

leads us to the one-form

$$\omega = A\,dx + B\,dy + C\,dz;$$

a surface integral

$$\iint P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

leads us to the two-form

$$\alpha = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy;$$

and a volume integral

$$\iiint H\,dx\,dy\,dz$$

leads us to the three-form

$$\lambda = H\,dx\,dy\,dz.$$

These are all examples of differential forms which live in the space $\mathbf{E}^3$ of three variables. If we work in an n-dimensional space, the quantity under the integral sign in an r-fold integral (integral over an r-dimensional variety) is an *r-form in n variables.*

In the expression α above, we notice the absence of terms in $dz\,dy$, $dx\,dz$, $dy\,dx$, which suggests symmetry or skew-symmetry. The further absence of terms $dx\,dx$, $\cdots$ strongly suggests the latter.

We shall set up a calculus of differential forms which will have certain inner consistency properties, one of which is the rule for changing variables in a multiple integral. Our integrals are always oriented integrals, hence we never take absolute values of Jacobians.

Consider

$$\iint A(x, y)\,dx\,dy$$

with the change of variable

$$\begin{cases} x = x(u, v) \\ y = y(u, v). \end{cases}$$

We have

$$\iint A(x, y)\, dx\, dy = \iint A[x(u, v), y(u, v)] \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

which leads us to write

$$dx\, dy = \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv.$$

If we set $y = x$, the determinant has equal rows, hence vanishes. Also if we interchange x and y, the determinant changes sign. This motivates the rules

$$\begin{cases} dx\, dx = 0 \\ dy\, dx = -dx\, dy \end{cases}$$

for multiplication of differentials in our calculus.

In general, an (*exterior*) *r-form in n variables* $x^1, \cdots, x^n$ will be an expression

$$\omega = \frac{1}{r!} \sum A_{i_1, \ldots, i_r}\, dx^{i_1} \cdots dx^{i_r},$$

where the coefficients A are smooth functions of the variables and skew-symmetric in the indices.

We shall associate with each r-form ω an $(r + 1)$-form $d\omega$ called the exterior derivative of ω. Its definition will be given in such a way that validates the general Stokes' formula

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega.$$

Here Σ is an $(r + 1)$-dimensional oriented variety and $\partial\Sigma$ is its boundary.

A basic relation is the Poincaré Lemma:

$$d\,(d\omega) = 0.$$

In all cases this reduces to the equality of mixed second partials.

1.2. Comparison with Tensors

At the outset we can assure our readers that we shall not do away with tensors by introducing differential forms. Tensors are here to stay; in a great many situations, particularly those dealing with symmetries, tensor methods are very natural and effective. However, in many other situations the use of the exterior calculus, often combined with the method of moving frames of É. Cartan, leads to decisive results in a way which is very difficult with tensors alone. Sometimes a combination of techniques is in order. We list several points of contrast.

(a) Tensor analysis *per se* seems to consist only of techniques for calculations with indexed quantities. It lacks a body of substantial or deep results established once and for all within the subject and then available for application. The exterior calculus does have such a body of results.

If one takes a close look at Riemannian geometry as it is customarily developed by tensor methods one must seriously ask whether the geometric results cannot be obtained more cheaply by other machinery.

(b) In classical tensor analysis, one never knows what is the range of applicability simply because one is never told what the space is. Everything seems to work in a coordinate patch, but we know this is inadequate for most applications. For example, if a particle is constrained to move on the sphere $\mathbf{S}^2$, a single coordinate system cannot describe its position space, let alone its phase or state spaces.

This difficulty has been overcome in modern times by the theory of differentiable manifolds (varieties) which we discuss in Chapter V.

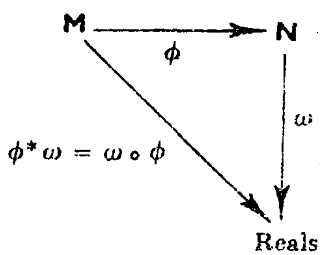
(c) Tensor fields do not behave themselves under mappings. For example, given a contravariant vector field a^i on x-space and a mapping ϕ on x-space to y-space, there is no naturally induced field on the y-space. [Try the map $t \longrightarrow (t^2, t^3)$ on $\mathbf{E}^1$ into $\mathbf{E}^2$.]

With exterior forms we have a really attractive situation in this regard. If

$$\phi: \quad \mathbf{M} \longrightarrow \mathbf{N}$$

and if ω is a p-form on $\mathbf{N}$, there is naturally induced a p-form $\phi^* \omega$ on $\mathbf{M}$.

Let us illustrate this for the simplest case in which ω is a 0-form, or scalar, i.e., a real-valued function on $\mathbf{N}$. Here $\phi^* \omega = \omega \circ \phi$, the composition of the mapping ϕ followed by ω.



(d) In tensor calculations the maze of indices often makes one lose sight of the very great differences between various types of quantities which can be represented by tensors, for example, vectors tangent to a space, mappings between such vectors, geometric structures on the tangent spaces.

(e) It is often quite difficult using tensor methods to discover the deeper invariants in geometric and physical situations, even the local ones. Using exterior forms, they seem to come naturally according to these principles:

(i) All local geometric relations arise one way or another from the equality of mixed partials, i.e., Poincaré's Lemma.

(ii) Local invariants themselves usually appear as the result of applying exterior differentiation to everything in sight.

(iii) Global relations arise from integration by parts, i.e., Stokes' theorem.

(iv) Existence problems which are not genuine partial differential equations (boundary value or Cauchy problems) generally are of the type of Frobenius–Cartan–Kähler system of exterior differential forms and can be reduced thereby to systems of ordinary equations.

(f) In studying geometry by tensor methods, one is invariably restricted to the *natural frames* associated with a local coordinate system. Let us consider a Riemannian geometry as a case in point. This consists of a manifold in which a Euclidean geometry has been imposed in each of the tangent spaces. A natural frame leads to an oblique coordinate system in each tangent space. Now who in his right mind would study Euclidean geometry with oblique coordinates? Of course the orthonormal coordinate systems are the natural ones for Euclidean geometry, so they must be the correct ones for the much harder Riemannian geometry. We are led to introduce moving frames, a method which goes hand-in-glove with exterior forms.

We conclude the case by stating our opinion, that exterior calculus is here to stay, that it will gradually replace tensor methods in numerous situations where it is the more natural tool, that it will find more and more applications because of its inner simplicity, body of substantial results begging for further use, and because it simply is there wherever integrals occur. There is generally a time lag of some fifty years between mathematical theories and their applications. The mathematicians H. Poincaré, É. Goursat, and É. Cartan developed the exterior calculus in the early part of this century; in the last twenty years it has greatly contributed to the rebirth of differential geometry, now part of the mathematical main stream. Physicists are beginning to realize its usefulness; perhaps it will soon make its way into engineering.

II

Exterior Algebra

2.1. The Space of p-Vectors

Notation:

$\mathbf{R}$ = field of real numbers $a, b, c, \cdots$.

$\mathbf{L}$ = an n-dimensional vector space over $\mathbf{R}$ with elements $\alpha, \beta, \cdots$.

For each $p = 0, 1, 2, \cdots, n$ we shall construct a new vector space

$$\bigwedge^p \mathbf{L}$$

over $\mathbf{R}$, called *the space of p-vectors on* $\mathbf{L}$. We begin with

$$\bigwedge^0 \mathbf{L} = \mathbf{R}, \qquad \bigwedge^1 \mathbf{L} = \mathbf{L}.$$

Next we shall work out $\bigwedge^2 \mathbf{L}$ in some detail. This space consists of all sums

$$\sum a_i(\alpha_i \wedge \beta_i)$$

subject only to these constraints, or reduction rules, and no others:

$$\begin{cases} (a_1\alpha_1 + a_2\alpha_2) \wedge \beta - a_1(\alpha_1 \wedge \beta) - a_2(\alpha_2 \wedge \beta) = 0, \\ \alpha \wedge (b_1\beta_1 + b_2\beta_2) - b_1(\alpha \wedge \beta_1) - b_2(\alpha \wedge \beta_2) = 0, \\ \alpha \wedge \alpha = 0, \\ \alpha \wedge \beta + \beta \wedge \alpha = 0. \end{cases}$$

Here α, β, etc., are vectors in $\mathbf{L}$ and a, b, etc., are real numbers; $\alpha \wedge \beta$ is called the *exterior product* of the vectors α and β. If α and β are dependent, say $\beta = c\alpha$, then

$$\alpha \wedge \beta = \alpha \wedge (c\alpha) = c(\alpha \wedge \alpha) = c \cdot 0 = 0$$

according to our reductions. Otherwise $\alpha \wedge \beta \neq 0$.

Suppose $\sigma^1, \cdots, \sigma^n$ is a basis of $\mathbf{L}$. Then

$$\alpha = \sum a_i\sigma_i, \qquad \beta = \sum b_j\sigma^j,$$

$$\alpha \wedge \beta = \left(\sum a_i\sigma^i\right) \wedge \left(\sum b_j\sigma^j\right) = \sum a_ib_j(\sigma^i \wedge \sigma^j).$$

We rearrange this as follows. Each term $\sigma^i \wedge \sigma^i = 0$ and each $\sigma^j \wedge \sigma^i = -\sigma^i \wedge \sigma^j$ for $i < j$. Hence

$$\alpha \wedge \beta = \sum_{i<j} (a_ib_j - a_jb_i)\sigma^i \wedge \sigma^j.$$

The typical element of $\bigwedge^2 \mathbf{L}$ is a linear combination of such exterior products, hence the 2-vectors

$$\sigma^i \wedge \sigma^j, \qquad 1 \leqq i < j \leqq n,$$

form a basis of $\bigwedge^2 \mathbf{L}$. We conclude

$$\dim \bigwedge^2 \mathbf{L} = \frac{n(n-1)}{2} = \binom{n}{2}.$$

In general, we form $\bigwedge^p \mathbf{L}$ $(2 \leqq p \leqq n)$ by the same idea. It consists of all formal sums (*p-vectors*, or vectors of degree p)

$$\sum a(\alpha_1 \wedge \cdots \wedge \alpha_p)$$

subject only to these constraints:

(i) $(a\alpha + b\beta) \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$
$$= a(\alpha \wedge \alpha_2 \wedge \cdots \wedge \alpha_p) + b(\beta \wedge \alpha_2 \wedge \cdots \wedge \alpha_p),$$
and the same if any α_i is replaced by a linear combination.

(ii) $\alpha_1 \wedge \cdots \wedge \alpha_p = 0$ if for some pair of indices $i \neq j$, $\alpha_i = \alpha_j$.

(iii) $\alpha_1 \wedge \cdots \wedge \alpha_p$ changes sign if any two α_i are interchanged.

It follows easily from (i) that $\alpha_1 \wedge \cdots \wedge \alpha_p$ is linear in each variable; we may replace any variable by a linear combination of any number (not just two) of other vectors and compute the value by distributing, for example

$$\alpha \wedge (b_1\beta_1 + b_2\beta_2 + b_3\beta_3) \wedge \gamma \wedge \delta$$
$$= b_1(\alpha \wedge \beta_1 \wedge \gamma \wedge \delta) + b_2(\alpha \wedge \beta_2 \wedge \gamma \wedge \delta) + b_3(\alpha \wedge \beta_3 \wedge \gamma \wedge \delta).$$

It follows from (iii) that if π is any permutation of the $\{1, 2, \cdots, p\}$, then

$$\alpha_{\pi(1)} \wedge \cdots \wedge \alpha_{\pi(p)} = (\operatorname{sgn} \pi)\alpha_1 \wedge \cdots \wedge \alpha_p.$$

Exactly as in the case $p = 2$, we can show that if

$$\sigma^1, \cdots, \sigma^n$$

is a basis of $\mathbf{L}$, then a basis of $\bigwedge^p \mathbf{L}$ is made up as follows. for each set of indices

$$H = \{h_1, h_2, \cdots, h_p\}, \qquad 1 \leqq h_1 < h_2 < \cdots < h_p \leqq n,$$

we set

$$\sigma^H = \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}.$$

Then the totality of σ^H is a basis of $\bigwedge^p \mathbf{L}$. We conclude that

$$\dim \bigwedge^p \mathbf{L} = \binom{n}{p},$$

the number of combinations of n things taken p at a time. In particular

$$\dim \bigwedge^n \mathbf{L} = 1.$$

If λ is in $\bigwedge^p \mathbf{L}$, then

$$\lambda = \sum_H a_H \sigma^H,$$

summed over all of these ordered sets H. One can also sum over all p-tuples of indices by introducing skew-symmetric coefficients:

$$\lambda = \frac{1}{p!} \sum_{h_1, \cdots, h_p} b_{h_1, \cdots, h_p} \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}$$

where the $b_{h_1 \cdots h_p}$ is a skew-symmetric tensor and

$$b_{h_1 \cdots h_p} = a_H \qquad \text{for} \qquad H = \{h_1, \cdots, h_p\}, \qquad h_1 < h_2 < \cdots < h_p.$$

This skew-symmetric representation is often quite useful.

Let us note why we do not define $\bigwedge^p \mathbf{L}$ for $p > n$. (Sometimes it is convenient to simply set $\bigwedge^p \mathbf{L} = 0$ for $p > n$.) We express each α in a product $\alpha_1 \wedge \cdots \wedge \alpha_p$ as a linear combination of the basis vectors $\sigma^1, \cdots, \sigma^n$ and completely distribute according to Rule (i). This leads to

$$\alpha_1 \wedge \cdots \wedge \alpha_p = \sum a_{h_1 \cdots h_p} \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}.$$

Each term $\sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}$ is a product of $p > n$ vectors taken from the set $\sigma^1, \cdots, \sigma^n$ so there must be a repetition; by Rule (ii) it vanishes. We are left with $\alpha_1 \wedge \cdots \wedge \alpha_p = 0$ as the only possibility.

We close with a very important property of the spaces $\bigwedge^p \mathbf{L}$.

In order to define a linear mapping f on $\bigwedge^p \mathbf{L}$ it suffices to present a function g of p variables on $\mathbf{L}$ such that (i) g is linear in each variable separately, (ii) g is alternating in the sense that g vanishes when two of its variables are equal and g changes sign when two of its variables are interchanged. Then

$$f(\alpha_1 \wedge \cdots \wedge \alpha_p) = g(\alpha_1, \cdots, \alpha_p)$$

defines f on the generators of $\bigwedge^p \mathbf{L}$.

It can be shown that this property provides an axiomatic characterization of $\bigwedge^p \mathbf{L}$. In the next section we apply this property to define the determinant of a linear transformation.

2.2. Determinants

As above $\mathbf{L}$ is a fixed linear space of dimension n. Let A be a linear transformation on $\mathbf{L}$ into itself. We define a function $g = g_A$ of n variables on $\mathbf{L}$ as follows:

$$g_A(\alpha_1, \cdots, \alpha_n) = A\alpha_1 \wedge \cdots \wedge A\alpha_n,$$

$$g_A\colon \ \bigtimes^n \mathbf{L} \longrightarrow \bigwedge^n \mathbf{L}$$

where $\times^n \mathbf{L}$ denotes the cartesian product. Since g is multilinear and alternating, there is a linear functional $f = f_A$,

$$f_A\colon \quad \bigwedge^n \mathbf{L} \longrightarrow \bigwedge^n \mathbf{L}$$

satisfying

$$f_A(\alpha_1 \wedge \cdots \wedge \alpha_n) = g_A(\alpha_1, \cdots, \alpha_n) = A\alpha_1 \wedge \cdots \wedge A\alpha_n.$$

But $\bigwedge^n \mathbf{L}$ is one-dimensional so the only linear transformation on this space is multiplication by a scalar. We denote the particular one here by $|A|$ and have

$$A\alpha_1 \wedge \cdots \wedge A\alpha_n = |A|(\alpha_1 \wedge \cdots \wedge \alpha_n).$$

This serves to define the *determinant* $|A|$ of A. We must not fail to note that this definition is completely independent of a matrix representation of A.

We observe next

$$\begin{aligned} |AB|(\alpha_1 \wedge \cdots \wedge \alpha_n) &= (AB\alpha_1) \wedge \cdots \wedge (AB\alpha_n) \\ &= |A|(B\alpha_1 \wedge \cdots \wedge B\alpha_n) \\ &= |A| \cdot |B|(\alpha_1 \wedge \cdots \wedge \alpha_n), \end{aligned}$$

hence

$$|AB| = |A| \cdot |B|.$$

We can relate this to the determinant of a matrix as follows. Let $\sigma^1, \cdots, \sigma^n$ be a basis of $\mathbf{L}$ and $\|a_{ij}\|$ an $n \times n$ matrix. Set

$$\alpha_i = \sum a_{ij}\sigma^j.$$

Then

$$\alpha_1 \wedge \cdots \wedge \alpha_n = |a_{ij}|\sigma^1 \wedge \cdots \wedge \sigma^n.$$

In particular, if one obtains the matrix representation of A with respect to the basis (σ^i) by

$$A\sigma^i = \sum a^i{}_j \sigma^j,$$

then $\quad A\sigma^1 \wedge \cdots \wedge A\sigma^n = |a^i{}_j|\sigma^1 \wedge \cdots \wedge \sigma^n, \qquad |A| = |a^i{}_j|.$

2.3. Exterior Products

We now observe that our spaces $\bigwedge^p \mathbf{L}$ have a built-in multiplication process called *exterior multiplication* and denoted by $\wedge$ for obvious reasons. We multiply a p-vector μ by a q-vector ν to obtain a $(p+q)$-vector $\mu \wedge \nu$ (which is 0 by definition if $p + q > n$):

$$\wedge\colon \quad (\bigwedge^p \mathbf{L}) \times (\bigwedge^q \mathbf{L}) \longrightarrow \bigwedge^{p+q} \mathbf{L}.$$

It suffices to define $\wedge$ on generators and use the basic principle at the end of Section 1 to extend it to all p- and q-vectors:

$$(\alpha_1 \wedge \cdots \wedge \alpha_p) \wedge (\beta_1 \wedge \cdots \wedge \beta_q) = \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q.$$