Graduate Texts in Mathematics

R. E. Edwards

Fourier Series
A Modern Introduction
Volume 1

Second Edition

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PREFACE

The principal aim in writing this book has been to provide an introduction, barely more, to some aspects of Fourier series and related topics in which a liberal use is made of modern techniques and which guides the reader toward some of the problems of current interest in harmonic analysis generally. The use of modern concepts and techniques is, in fact, as widespread as is deemed to be compatible with the desire that the book shall be useful to senior undergraduates and beginning graduate students, for whom it may perhaps serve as preparation for Rudin's *Harmonic Analysis on Groups* and the promised second volume of Hewitt and Ross's *Abstract Harmonic Analysis*.

The emphasis on modern techniques and outlook has affected not only the type of arguments favored, but also to a considerable extent the choice of material. Above all, it has led to a minimal treatment of pointwise convergence and summability: as is argued in Chapter 1, Fourier series are not necessarily seen in their best or most natural role through pointwise-tinted spectacles. Moreover, the famous treatises by Zygmund and by Bary on trigonometric series cover these aspects in great detail, while leaving some gaps in the presentation of the modern viewpoint; the same is true of the more elementary account given by Tolstov. Likewise, and again for reasons discussed in Chapter 1, trigonometric series in general form no part of the program attempted.

A considerable amount of space has been devoted to matters that cannot in a book of this size and scope receive detailed treatment. Among such material, much of which appears in small print, appear comments on diverse specialized topics (capacity, spectral synthesis sets, Helson sets, and so forth), as well as remarks on extensions of results to more general groups. The object in including such material is, in the first case, to say enough for the reader to gain some idea of the meaning and significance of the problems involved, and to provide a guide to further reading; and in the second case, to provide some sort of "cultural" background stressing a unity that underlies apparently diverse fields. It cannot be over-emphasized that the book is perforce introductory in all such matters.

The demands made in terms of the reader's active cooperation increase fairly steadily with the chapter numbers, and although the book is surely best regarded as a whole, Volume I is self-contained, is easier than Volume II, and might be used as the basis of a short course. In such a short course, it would be feasible to omit Chapter 9 and Section 10.6.

As to specific requirements made of the reader, the primary and essential item is a fair degree of familiarity with Lebesgue integration to at least the extent described in Williamson's introductory book *Lebesgue Integration.* Occasionally somewhat more is needed, in which case reference is made to Appendix C, to Hewitt and Stromberg's *Real and Abstract Analysis*, or to Asplund and Bungart's *A First Course in Integration*. In addition, the reader needs to know what metric spaces and normed linear spaces are, and to have some knowledge of the rudiments of point-set topology. The remaining results in functional analysis (category arguments, uniform boundedness principles, the closed graph, open mapping, and Hahn-Banach theorems) are dealt with in Appendixes A and B. The basic terminology of linear algebra is used, but no result of any depth is assumed.

Exercises appear at the end of each chapter, the more difficult ones being provided with hints to their solutions.

The bibliography, which refers to both book and periodical literature, contains many suggestions for further reading in almost all relevant directions and also a sample of relevant research papers that have appeared since the publication of the works by Zygmund, Bary, and Rudin already cited. Occasionally, the text contains references to reviews of periodical literature.

My first acknowledgment is to thank Professors Hanna Neumann and Edwin Hewitt for encouragement to begin the book, the former also for the opportunity to try out early drafts of Volume I on undergraduate students in the School of General Studies of the Australian National University, and the latter also for continued encouragement and advice. My thanks are due also to the aforesaid students for corrections to the early drafts.

In respect to the technical side of composition, I am extremely grateful to my colleague, Dr. Garth Gaudry, who read the entire typescript (apart from last-minute changes) with meticulous care, made innumerable valuable suggestions and vital corrections, and frequently dragged me from the brink of disaster. Beside this, the compilation of Sections 13.7 and 13.8 and Subsection 13.9.1 is due entirely to him. Since, however, we did not always agree on minor points of presentation, I alone must take the blame for shortcomings of this nature. To him I extend my warmest thanks.

My thanks are offered to Mrs. Avis Debnam, Mrs. K. Sumeghy, and Mrs. Gail Liddell for their joint labors on the typescript.

Finally, I am deeply in debt to my wife for all her help with the proofreading and her unfailing encouragement.

R. E. E.

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67 Serre. Local Fields.
68 Weidmann. Linear Operators in Hilbert Spaces.

CONTENTS

CHAPTER 1

Trigonometric Series and Fourier Series

1.1 The Genesis of Trigonometric Series and Fourier Series

1.1.1. The Beginnings. D. Bernoulli, D'Alembert, Lagrange, and Euler, from about 1740 onward, were led by problems in mathematical physics to consider and discuss heatedly the possibility of representing a more or less arbitrary function f with period 2π as the sum of a *trigonometric series* of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1.1}$$

or of the formally equivalent series in its so-called "complex" form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1.1.1*}$$

in which, on writing $b_0 = 0$, the coefficients c_n are given by the formulae

$$c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \tfrac{1}{2}(a_n + ib_n) \qquad (n = 0, 1, 2, \cdots).$$

This discussion sparked off one of the crises in the development of analysis.

Fourier announced his belief in the possibility of such a representation in 1811. His book *Théorie Analytique de la Chaleur*, which was published in 1822, contains many particular instances of such representations and makes widespread heuristic use of trigonometric expansions. As a result, Fourier's name is customarily attached to the following prescription for the coefficients a_n, b_n, and c_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \tag{1.1.2}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx, \tag{1.1.2*}$$

the a_n and b_n being now universally known as the "real," and the c_n as the "complex," *Fourier coefficients* of the function f (which is tacitly assumed to be integrable over $(-\pi, \pi)$). The formulae (1.1.2) were, however, known earlier to Euler and Lagrange.

The grounds for adopting Fourier's prescription, which assigns a definite trigonometric series to each function f that is integrable over $(-\pi, \pi)$, will be scrutinized more closely in 1.2.3. The series (1.1.1) and (1.1.1*), with the coefficients prescribed by (1.1.2) and (1.1.2*), respectively, thereby assigned to f are termed the "real" and "complex" *Fourier series* of f, respectively.

During the period 1823–1827, both Poisson and Cauchy constructed proofs of the representation of restricted types of functions f by their Fourier series, but they imposed conditions which were soon shown to be unnecessarily stringent.

It seems fair to credit Dirichlet with the beginning of the rigorous study of Fourier series in 1829, and with the closely related concept of function in 1837. Both topics have been pursued with great vigor ever since, in spite of more than one crisis no less serious than that which engaged the attentions of Bernoulli, Euler, d'Alembert, and others and which related to the prevailing concept of functions and their representation by trigonometric series. (Cantor's work in set theory, which led ultimately to another major crisis, had its origins in the study of trigonometric series.)

1.1.2. The rigorous developments just mentioned showed in due course that there are subtle differences between trigonometric series which converge at all points and Fourier series of functions which are integrable over $(-\pi, \pi)$, even though there may be no obvious clue to this difference. For example, the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$$

converges everywhere; but, as will be seen in Exercise 7.7 and again in 10.1.6, it is not the Fourier series of any function that is (Lebesgue-)integrable over $(-\pi, \pi)$.

The theory of trigonometric series in general has come to involve itself with many questions that simply do not arise for Fourier series. For the express purpose of attacking such questions, many techniques have been evolved which are largely irrelevant to the study of Fourier series. It thus comes about that Fourier series may in fact be studied quite effectively without reference to general trigonometric series, and this is the course to be adopted in this book.

The remaining sections of this chapter are devoted to showing that, while Fourier series have their limitations, general trigonometric series have others no less serious; and that there are well-defined senses and contexts in which Fourier series are the natural and distinguished tools for representing functions in useful ways. Any reader who is prepared to accept without question the restriction of attention to Fourier series can pass from 1.1.3 to the exercises at the end of this chapter.

1.1.3. The Orthogonality Relations. Before embarking upon the discussion promised in the last paragraph, it is necessary to record some facts that provide the heuristic basis for the Fourier formulae (1.1.2) and (1.1.2*) and for whatever grounds there are for according a special role to Fourier series.

These facts, which result from straightforward and elementary calculations, are expressed in the following so-called *orthogonality relations* satisfied by the circular and complex exponential functions:

$$\left.\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\begin{cases}0 & (m\neq n,\ m\geqslant 0,\ n\geqslant 0)\\ \tfrac{1}{2} & (m=n>0),\\ 1 & (m=n=0)\end{cases}\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\begin{cases}0 & (m\neq n,\ m\geqslant 0,\ n\geqslant 0)\\ \tfrac{1}{2} & (m=n>0),\\ 0 & (m=n=0)\end{cases}\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\sin nx\,dx=0,\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{imx}\,e^{-inx}\,dx=\begin{cases}0 & (m\neq n)\\ 1 & (m=n);\end{cases}
\end{aligned}\right\}\quad(1.1.3)$$

in these relations m and n denote integers, and the interval $[-\pi, \pi]$ may be replaced by any other interval of length 2π.

1.2 Pointwise Representation of Functions by Trigonometric Series

1.2.1. Pointwise Representation. The general theory of trigonometric series was inaugurated by Riemann in 1854, since when it has been pursued with vigor and to the great enrichment of analysis as a whole. For modern accounts of the general theory, see [Z_1], Chapter IX and [$Ba_{1,2}$], Chapters XII–XV.

From the beginning a basic problem was that of representing a more or less arbitrary given function f defined on a period-interval I (say the interval $[-\pi, \pi]$) as the sum of at least one trigonometric series (1.1.1), together with a discussion of the uniqueness of this representation.

A moment's thought will make it clear that the content of this problem depends on the interpretation assigned to the verb "to represent" or, what comes to much the same thing, to the term "sum" as applied to an infinite series. Initially, the verb was taken to mean the pointwise convergence of the series at all points of the period interval to the given function f. With the passage of time this interpretation underwent modification in at least two ways. In the first place, the demand for convergence of the series to f at *all* points of the period-interval I was relaxed to convergence at *almost all* points of that interval. In the second place, convergence of the series to f at all or almost all points was weakened to the demand that the series be *summable* to f by one of several possible methods, again at all or almost all points. For the purposes of the present discussion it will suffice to speak of just one such summability method, that known after Cesàro, which consists of replacing the partial sums

$$s_0(x) = \tfrac{1}{2}a_0,$$
$$s_N(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad (N = 1, 2, \cdots) \quad (1.2.1)$$

of the series (1.1.1) by their arithmetic means

$$\sigma_N = \frac{s_0 + \cdots + s_N}{N+1} \qquad (N = 0, 1, 2, \cdots). \quad (1.2.2)$$

Thus we shall say that the series (1.1.1) is summable at a point x to the function f if and only if

$$\lim_{N\to\infty} \sigma_N(x) = f(x).$$

It will be convenient to group all these interpretations of the verb "to represent" under the heading of *pointwise representation* (everywhere or almost everywhere, by convergence or by summability, as the case may be) of the function f by the series (1.1.1).

In terms of these admittedly rather crude definitions we can essay a bird's-eye view of the state of affairs in the realm of pointwise representation, and in particular we can attempt to describe the place occupied by Fourier series in the general picture.

1.2.2. **Limitations of Pointwise Representation.** Although it is undeniably of great intrinsic interest to know that a certain function, or each member of a given class of functions, admits a pointwise representation by some trigonometric series, it must be pointed out without delay that this type of representation leaves much to be desired on the grounds of *utility*. A mode of representation can be judged to be successful or otherwise useful as a tool in subsequent investigations by estimating what standard analytical operations applied to the represented function can, via the representation, be expressed with reasonable simplicity in terms of the expansion coefficients a_n and b_n. This is, after all, one of the main reasons for seeking a representation in series form. Now it is a sad fact that pointwise representations are in themselves not very useful in this sense; they are simply too weak to justify the termwise application of standard analytical procedures.

Another inherent defect is that a pointwise representation at almost all points of I is never unique. This is so because, as was established by Men'shov in 1916, there exist trigonometric series which converge to zero almost everywhere and which nevertheless have at least one nonvanishing coefficient; see 12.12.8. (That this can happen came as a considerable surprise to the mathematical community.)

1.2.3. The Role of the Orthogonality Relations. The a priori grounds for expecting the Fourier series of an integrable function f to effect a pointwise representation of f (or, indeed, to effect a representation in any reasonable sense) rest on the orthogonality relations (1.1.3). It is indeed a simple consequence of these relations that, if there exists *any* trigonometric series (1.1.1) which represents f in the pointwise sense, and if furthermore the s_N (or the σ_N) converge dominatedly (see [W], p. 60) to f, then the series (1.1.1) must be the Fourier series of f. However, the second conditional clause prevents any very wide-sweeping conclusions being drawn at the outset.

As will be seen in due course, the requirements expressed by the second conditional clause are fulfilled by the Fourier series of sufficiently smooth functions f (for instance, for those functions f that are continuous and of bounded variation). But, alas, the desired extra condition simply does not obtain for more general functions of types we wish to consider in this book. True, a greater degree of success results if convergence is replaced by summability (see 1.2.4). But in either case the investigation of this extra condition itself carries one well into Fourier-series lore. This means that this would-be simple and satisfying explanation for according a dominating role to Fourier series can scarcely be maintained at the *outset* for functions of the type we aim to study.

1.2.4. Fourier Series and Pointwise Representations. What has been said in 1.2.3 indicates that Fourier series can be expected to have but limited success in the pointwise representation problem. Let us tabulate a little specific evidence.

The Fourier series of a periodic function f which is continuous and of bounded variation converges boundedly at all points to that function. The Fourier series of a periodic continuous function may, on the contrary, diverge at infinitely many points; even the pointwise convergence almost everywhere of the Fourier series of a general continuous function remained in doubt until 1966 (see 10.4.5), although it had been established much earlier and much more simply that certain fixed subsequences of the sequence of partial sums of the Fourier series of any such function is almost everywhere convergent to that function (the details will appear in Section 8.6). The Fourier series of an integrable function may diverge at all points.

If ordinary convergence be replaced by summability, the situation improves. The Fourier series of a periodic continuous function is uniformly summable to that function. The Fourier series of any periodic integrable function is summable at almost all points to that function, but in this case neither the s_N nor the σ_N need be dominated.

1.2.5. **Trigonometric Series and Pointwise Representations.** Having reviewed a few of the limitations of Fourier series vis-à-vis the problem of pointwise representation, we should indicate what success is attainable by using trigonometric series in general.

In 1915 both Lusin and Privalov established the existence of a pointwise representation almost everywhere by summability methods of any function f which is measurable and finite almost everywhere. For 25 years doubts lingered as to whether summability could here be replaced by ordinary convergence, the question being resolved affirmatively by Men'shov in 1940. This result was sharpened in 1952 by Bary, who showed that, if the function f is measurable and finite almost everywhere on the interval I, there exists a continuous function F such that $F'(x) = f(x)$ at almost all points of I, and such that the series obtained by termwise differentiation of the Fourier series of F converges at almost all points x of I to $f(x)$. Meanwhile Men'shov had in 1950 shown also that to any measurable f (which may be infinite on a set of positive measure) corresponds at least one trigonometric series (1.1.1) whose partial sums s_N have the property that $\lim_{N\to\infty} s_N = f$ in *measure* on I. This means that one can write $s_N = u_N + v_N$, where u_N and v_N are finite-valued almost everywhere, $\lim_{N\to\infty} u_N(x) = f(x)$ at almost all points x of I, and where, for any fixed $\varepsilon > 0$, the set of points x of I for which $|v_N(x)| > \varepsilon$ has a measure which tends to zero as $N \to \infty$. (The stated condition on the v_N is equivalent to the demand that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{|v_N|\, dx}{1 + |v_N|} = 0;$$

and the circuitous phrasing is necessary because f may take infinite values on a set of positive measure.) This sense of representation is weaker than pointwise representation. For more details see [Ba_2], Chapter XV.

These theorems of Men'shov and Bary lie very deep and represent enormous achievements. However, as has been indicated at the end of 1.2.2, the representations whose existence they postulate are by no means unique.

Cantor succeeded in showing that a representation at *all* points by a convergent trigonometric series is necessarily unique, if it exists at all. Unfortunately, only relatively few functions f admit such a representation: for instance, there are continuous periodic functions f that admit no such representation. (This follows on combining a theorem due to du Bois-Reymond and Lebesgue, which appears on p. 202 of [Ba_1], with results about Fourier series dealt with in Chapter 10 of this book.) It is indeed the case that, in a sense, "most" continuous functions admit no representation of this sort.