Hyperbolic Differential Equations

by

Jean Leray

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HYPERBOLIC DIFFERENTIAL EQUATIONS

by

Jean Leray

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First part

Linear hyperbolic equations with constant coefficients and symbolic calculus with several variables.

Introduction

Since Hadamard gave his Yale lectures in 1920 about the hyperbolic differential equations of second order, many important papers about the equations of any order have been published by Herglotz, Schauder, Petrowsky, Bureau, M. Riesz, and Gårding. They used various but interesting methods and their results are important but incomplete.

The first part of our lectures is related to the equations with constant coefficients; it goes beyond Petrowsky's and Gårding's results, and it improves their methods. Hadamard and Bureau used the Green's formula; by means of a duality it transforms a boundary value problem into the problem of finding a particular solution with a given singularity; this transformed problem is easy and this particular solution is handy and important when the given problem is very simple; but generally this method is a difficult one. Herglotz, Petrowsky and Gårding use another duality: that between the independent variables and the derivations, which gives rise to the Fourier and Laplace transformation. More precisely Herglotz and Petrowsky applied the Heaviside calculus, that is to say the Laplace transformation <u>to one</u> variable (as a matter of fact to the time) and the Fourier-transformation <u>to the others</u> (as a matter of fact to the space); how shocking in a relativistic world! It is not astonishing that Gårding obtained more complete results by applying the Laplace transformation at once to all the variables. But he did not express all the results this transformation gives; for instance: he uses the director cones Γ of the convex domains Δ, without studying these important convex domains Δ; he defines some operators by means of the Laplace transformation and the others by means of the Riesz's analytical prolongation, whereas it is convenient to define and to study these operators all together by means of the Laplace transformation.

We do not transform any boundary problem; but by the Laplace transformation Chapter I defines and studies throughout the symbolic calculus of several variables; this calculus enables us to solve the Cauchy's boundary value problem for differential equations (and would also enable one to solve equations containing both derivatives and finite differences): see Chap. VII, §4, no. 107-108-109 and Chap. VIII, no. 113.

Chapter II gives in particular a new, general and concise expression of the inverse of $\frac{\partial^2}{\partial x_1^{\,2}} - \frac{\partial^2}{\partial x_2^{\,2}} - \dots - \frac{\partial^2}{\partial x^{\,2}}$ and of its powers.

Chapter III studies the inverse of any polynomial of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x}$.

Chapter IV, assuming that this polynomial is homogeneous, achieves Herglotz-Petrowsky's calculation of its elementary solution.

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CHAPTER I

THE SYMBOLIC CALCULUS

§1. Fourier and Laplace transforms

(This whole §1 is classical: see Bochner's books [13] and [14].)

1. Fourier transforms. (See the bibliography: [13], [15], [21], [22], [23], [25].) Let X and Ξ be two vector spaces of the same finite dimension ℓ over the field of real numbers. Suppose they are dual: there is given a bilinear function

$$x \cdot \eta \quad \text{of } x \in X \text{ and } \eta \in \Xi,$$

the values of which are real and which satisfies: if $x \cdot \eta = 0$ for some x and all η, then $x = 0$; if $x \cdot \eta = 0$ for some η and all x, then $\eta = 0$. (See: Bourbaki, Algèbre linéaire.)

The coordinates $x_1, \ldots x_\ell$ of x and $\eta_1 \ldots \eta_\ell$ of η are real numbers and will be so chosen that

$$x \cdot \eta = x_1 \eta_1 + \cdots + x_\ell \eta_\ell .$$

Let f(x), g(x) be functions with complex values, defined on X; let $\phi(\eta)$, $\psi(\eta)$ be functions with complex values, defined on Ξ; f(x)g(x) is the product of f(x) and g(x); $f(x)*g(x) = \int_X \cdots \int f(x-y)g(y)dy_1 \ldots dy_\ell$ is the convolution of f(x) and g(x). We use the norms

$$\|f(x)\|_q = \left[\int_X \cdots \int |f(x)|^q dx_1 \ldots dx_\ell\right]^{\frac{1}{q}} \text{ for } q = 1 \text{ or } q = 2$$

and

$$\|f(x)\|_\infty = \operatorname{Sup.}_X |f(x)|.$$

We then have

$$(1.1) \qquad (\|f\|_2)^2 \leq \|f\|_1 \cdot \|f\|_\infty .$$

It is easy to prove the following:

If $\|f\|_1 < +\infty$ and $\|g\|_1 < +\infty$, then $\|f*g\|_1 \leq \|f\|_1 \|g\|_1 < +\infty$; and

If $\|f\|_1 < +\infty$ and $\|g\|_2 < +\infty$, then $\|f*g\|_2 \leq \|f\|_1 \|g\|_2 < +\infty$.

Thus, by use of the convolution, the $f(x)$ such that $\|f\|_1 < +\infty$ constitute a ring and the $f(x)$ such that $\|f\|_2 < +\infty$ constitute a vector space over this ring.

If $\|f(x)\|_1 < +\infty$, then its Fourier transform $\mathcal{F}[f(x)]$ is the continuous and bounded function $\phi(\eta)$, which is given by the formula [exp. λ = e^λ]:

$$(1.2) \quad \mathcal{F}[f(x)] = \phi(\eta) = \int \ldots \int_X f(x) \exp.(-2\pi i x \cdot \eta)\, dx_1 \ldots dx_\ell .$$

It can be proved that, if $\|f\|_1$ and $\|g\|_1 < +\infty$, then $\mathcal{F}[g*f] = \mathcal{F}[g] \cdot \mathcal{F}[f]$, afterwards that $\|f(x)\|_2 = \|\phi(\eta)\|_2$ if $\phi = \mathcal{F}[f]$: this enables one to define $\mathcal{F}[f]$ whenever $\|f\|_2 < +\infty$. Plancherel proved the following regarding this extension:

$\mathcal{F}[f]$ is an isometric linear mapping of the Hilbert space of all functions $f(x)$ such that $\|f\|_2 < +\infty$ onto the Hilbert space of all functions $\phi(\eta)$ such that $\|\phi(\eta)\|_2 < +\infty$. (thus it is a unitary mapping.) This mapping is given by (1.2) whenever both $\|f\|_1$ and $\|f\|_2 < +\infty$; its inverse is given by

$$(1.3) \quad \mathcal{F}^{-1}[\phi(\eta)] = f(x) = \int \ldots \int_\Xi \phi(\eta) \exp.(2\pi i x \cdot \eta)\, d\eta_1 \ldots d\eta_\ell$$

whenever both $\|\phi\|_1$ and $\|\phi\|_2 < +\infty$.

We have also the following formulas:

(1.4) $\mathcal{F}[g * f] = \mathcal{F}[g]\mathcal{F}[f]$ if $\|g\|_1 < +\infty$ and $\|f\|_1$ or $\|f\|_2 < +\infty$;

likewise

(1.5) $\mathcal{F}[gf] = \mathcal{F}[g] * \mathcal{F}[f]$ if $\|g\|_2$ and $\|f\|_2 < +\infty$;

further, if $\mathcal{F}[f] = \phi$, then:

(1.6) $\mathcal{F}[f(Sx)] = \phi(\Sigma\eta)|\det.\Sigma|$ ($\det.\Sigma$ = determinant of Σ)

for any couple S, Σ of <u>contragredient</u> linear mappings of X and Ξ (that is to say: $x \cdot \eta = Sx \cdot \Sigma\eta$ for any $x \in X$ and $\eta \in \Xi$)

(1.7) $\mathcal{F}[f(x + y)] = \phi(\eta)\exp.(2\pi i y \cdot \eta)$ for any $y \in X$;

(1.8) $\mathcal{F}[f(x)\exp.(2\pi i x \cdot \xi)] = \phi(\eta - \xi)$ for any $\xi \in \Xi$;

(1.9) $$\mathcal{F}[x_1 f(x)] = \frac{i}{2\pi}\frac{\partial\phi}{\partial\eta_1};$$

(1.10) $$\mathcal{F}\left[\frac{\partial f}{\partial x_1}\right] = 2\pi i\eta_1\phi(\eta).$$

<u>Remark</u> 1.1. There is an easy extension of formula (1.5):

(1.11) If $\mathcal{F}[f(x, x')] = \phi(\eta, \eta')$, where x and $x' \in X$, η and $\eta' \in \Xi$,

then $\mathcal{F}[f(x, x)] = \int \cdots \int_{\Xi} \phi(\eta - \eta', \eta')d\eta'_1 \ldots d\eta'_\ell$.

<u>Remark</u> 1.2. The formula (1.10) shows that the Fourier transformation reduces the solution of a differential equation with constant coefficients to division by a polynomial ... if the solution of the differential equation has a Fourier transform, which happens rarely; therefore it is necessary to use the closely related Laplace transforms.

2. <u>Laplace transformations</u>. The Laplace transform of $f(x)$ is the function of $\zeta = \xi + i\eta$ ($\xi \in \Xi$, $\eta \in \Xi$, $i = \sqrt{-1}$; $\zeta_1 = \xi_1 + i\eta_1$, $\ldots$, $\zeta_\ell = \xi_\ell + i\eta_\ell$)

(2.1) $$\mathcal{L}[f(x)] = \phi(\zeta) = \mathcal{F}[f(x) \exp.(-2\pi x \cdot \xi)].$$

If $\| f(x) \exp.(-2\pi x \cdot \xi)\|_1 < +\infty$ for some ξ, then

(2.2) $$\mathcal{L}[f] = \int_X \cdots \int f(x) \exp.(-2\pi x \cdot \zeta)\, dx_1 \ldots dx_\ell ;$$

$\mathcal{L}[f]$ is also defined if $\| f(x) \exp.(-2\pi x \cdot \xi)\|_2 < +\infty$ for some ξ. If $\| \phi(\xi + i\eta)\|_2 < +\infty$ for some fixed ξ, then $\mathcal{L}^{-1}[\phi]$ exists [but it could depend on ξ: see n°4]; if further $\| \phi(\xi + i\eta)\|_1 < +\infty$, then

(2.3) $$\mathcal{L}^{-1}[\phi] = \int_\Xi \cdots \int \phi(\xi + i\eta) \exp.[2\pi x \cdot (\xi + i\eta)]\, d\eta_1 \ldots d\eta_\ell .$$

The formulas of n°1 give the following ones upon application of (2.1):

(2.4) $\mathcal{L}[g * f] = \mathcal{L}[g]\mathcal{L}[f]$ if for some ξ, $\| g(x) \exp.(-2\pi x \cdot \xi)\|_1 < +\infty$ and $\| f(x) \exp.(-2\pi x \cdot \xi)\|_1$ or $\| f(x) \exp.(-2\pi x \cdot \xi)\|_2 < +\infty$;

(2.5) $\mathcal{L}[gf] = \mathcal{L}[g] * \mathcal{L}[f]$, if for some ξ and $\xi' \in \Xi$, $\| g(x) \exp.[-2\pi x \cdot (\xi - \xi')]\|_2 < +\infty$, $\| f(x) \exp.(-2\pi x \cdot \xi')\|_2 < +\infty$;

$\phi(\zeta) * \psi(\zeta)$ means $\int_\Xi \cdots \int \phi(\xi + i\eta - \xi' - i\eta')\, \psi(\xi' + i\eta')\, d\eta'_1 \ldots d\eta'_\ell$;

(2.6) $$\mathcal{L}[f(Sx)] = \phi(\Sigma\zeta)\,|\det.\Sigma|$$

for any couple S, Σ of contragredient linear mappings of X and Ξ;

(2.7) $$\mathcal{L}[f(x + y)] = \phi(\zeta) \exp.(2\pi y \cdot \zeta) \text{ for any } y \in X;$$

(2.8) $\mathcal{L}[f(x) \exp.(2\pi x \cdot \zeta')] = \phi(\zeta - \zeta')$ if $\zeta' \in \Xi + i\,\Xi$;

(2.9) $$\mathcal{L}[x_1 f(x)] = -\frac{1}{2\pi}\frac{\partial \phi(\zeta)}{\partial \zeta_1}$$

$$(2.10) \qquad \mathcal{L}\left[\frac{\partial f}{\partial x_1}\right] = 2\pi\, \zeta_1\, \phi(\zeta)\,.$$

Remark 2.1. The second part of these lectures will use the extension of formula (2.5) which follows from (1.11):

(2.11) If $\mathcal{L}[f(x,x')] = \phi(\zeta, \zeta')$, where x and $x' \in X$, ζ and $\zeta' \in \Xi + i\Xi$

$$\text{then } \mathcal{L}[f(x,x)] = \int \dots \int_{\Xi} \phi(\zeta - \xi', \zeta')\, d\eta'_1 \dots d\eta'_\ell.$$

Note. n°4 studies $\mathcal{L}^{-1}$, using the definitions given in n°3.

3. The ring of distributions E(Δ) and the subring of functions F(Δ).

Proposition 3.1. $\log \| f(x) \exp.(-x \cdot \xi) \|_q$ is a convex function of ξ; therefore the set on which it is finite is convex.

Proof. Let ξ and η be two points of Ξ; let μ and ν be two positive numbers such that $\mu + \nu = 1$; the classical Hölder's inequality (see [13], ch. III, L_p-spaces, §5)

$$(3.1) \qquad \| f \cdot g \|_q \leq \| f \|_{q/\mu}\, \| g \|_{q/\nu}$$

gives

$$\| f(x) \exp.(-x \cdot \mu\xi - x \cdot \nu\eta) \|_q =$$

$$\| \, |f(x) \exp.(-x \cdot \xi)|^\mu \, |f(x) \exp.(-x \cdot \eta)|^\nu \, \|_q \leq$$

$$\| \, |f(x) \exp.(-x \cdot \xi)|^\mu \, \|_{q/\mu} \, \| \, |f(x) \exp.(-x \cdot \eta)|^\nu \, \|_{q/\nu} =$$

$$[\, \|f(x) \exp.(-x \cdot \xi)\|_q]^\mu \cdot [\, \|f(x) \exp.(-x \cdot \eta)\|_q]^\nu$$

Definition of Δ. Besides X and Ξ a convex domain Δ of Ξ is given. [A domain is an open and connected set].

Definition of F(Δ). F(Δ) is the set of functions f(x) defined on X and such that $\| f(x) \exp.(- \xi \cdot x) \|_2 < + \infty$ for any $\xi \in \Delta$.

Remark 3.1. The proposition 3.1 proves that, if Δ had not been supposec convex, then $F(\Delta)$ would not change by replacing Δ by its convex closure.

Remark 3.2. This proposition 3.1 proves also that $\| f(x) \exp.(- x \cdot \xi) \|$ is uniformly bounded in Δ (that means: on any compact subset of Δ).

Lemma 3.1. If $f(x) \in F(\Delta)$ and $\xi \in \Delta$, then $\| f(x) \exp.(- x \cdot \xi) \|_1 < + \infty$ (it is uniformly bounded in Δ).

Proof. It is sufficient to prove that $\| f(x) \|_1$ is finite for $0 \in \Delta$ and $f(x) = 0$ outside the domain $x_1 > 0, x_2 > 0, \ldots x_\ell > 0$. Let $h(x)$ be the function equal to 1 in this domain and 0 outside; (3.1) gives $(\| f(x) \|_1)^2 \leq \| f(x) \exp.[\mathcal{E}(x_1 + \ldots + x_\ell)] \|_2 \cdot \| h(x) \exp.[- \mathcal{E}(x_1 + \ldots + x_\ell)] \|_2$ where $\mathcal{E}$ is a positive number, so small that $(- \mathcal{E}, \ldots - \mathcal{E}) \in \Gamma$.

This lemma and the formula (n°1) $\| f*g \|_2 \leq \| f \|_1 \| g \|_2$ prove that $f*g \in F(\Delta)$ if f and $g \in F(\Delta)$; thus

Proposition 3.2. $F(\Delta)$ is, by use of convolution, a ring.

Now let us use the Schwartz's theorie of distributions [21], which makes it possible to derive any function: the derivatives so obtained are functions or distributions.

Definition of $E(\Delta)$. The functions $f(x) \in F(\Delta)$ have derivatives of all orders; their convolutions are derivatives of functions of $F(\Delta)$, since $F(\Delta)$ is a ring. Therefore the finite sums of derivatives of elements of $F(\Delta)$ constitute a ring $E(\Delta)$; (the product to be used in this ring is the convolution). In other words: $E(\Delta)$ is the smallest ring which contains $F(\Delta)$ and is stable for the derivation.

4. The ring of analytic functions $\mathcal{L}[E]$ and its ideal A. Now we look

at the space $\Xi + i\,\Xi$: it is a vector space on the ring of complex numbers its points are the points

$$\zeta = \xi + i\eta \quad (\xi = \text{real part of } \zeta;\ \eta = \text{imaginary part of } \zeta),$$

where $\xi \in \Xi$, $\eta \in \Xi$. The <u>tube with basis</u> Δ is the set of points $\zeta = \xi + i\eta$ such that $\xi \in \Delta$ (Bochner; see [14], ch. V, § 4, p. 90). Let the function $a(\zeta)$ be analytic in this tube; we denote

$$\|a(\xi + i\eta)\|_q = \Big[\int_{\Xi}\!\cdots\!\int |a(\xi + i\eta)|^q \, d\eta_1 \ldots d\eta_\ell\Big]^{\frac{1}{q}};$$

$$\|a(\xi + i\eta)\|_\infty = \sup_{\eta \in \Xi} |a(\xi + i\eta)|.$$

The Cauchy's formula

$$a(\zeta) = \frac{1}{(2\pi)^\ell}\int_0^{2\pi}\!\cdots\!\int_0^{2\pi}$$

$$a[\zeta_1 + \rho_1 \exp.(i\phi_1), \ldots \zeta_\ell + \rho_\ell \exp.(i\phi_\ell)]\, d\phi_1 \ldots d\phi_\ell$$

enables us to express $a(\zeta)$ by an integral on a neighborhood of ζ, which proves this:

<u>Lemma</u> 4.1. If $\|a(\xi + i\eta)\|_q$ is bounded on any compact subset of Δ, then $\|a(\xi + i\eta)\|_\infty$ has the same property.

The Plancherel's theorem (n°1) has the following consequence [see the same proposition and a similar proof by [14], ch. VI, § 8, p. 128].

<u>Proposition</u> 4.1. <u>$\mathcal{L}$ maps $F(\Delta)$ one-one on the set $\Phi(\frac{1}{2\pi}\Delta)$ of the functions $\phi(\zeta)$ which are analytic in the tube with basis $\frac{1}{2\pi}\Delta$ and are such that $\|\phi(\xi + i\eta)\|_2$ is uniformly bounded in $\frac{1}{2\pi}\Delta$.</u>

<u>Moreover</u>