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Frank W. Anderson
Kent R. Fuller

Rings and Categories of Modules

2nd Edition

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Preface

This book is intended to provide a reasonably self-contained account of a major portion of the general theory of rings and modules suitable as a text for introductory and more advanced graduate courses. We assume the familiarity with rings usually acquired in standard undergraduate algebra courses. Our general approach is categorical rather than arithmetical. The continuing theme of the text is the study of the relationship between the one-sided ideal structure that a ring may possess and the behavior of its categories of modules.

Following a brief outline of set-theoretic and categorical foundations, the text begins with the basic definitions and properties of rings, modules and homomorphisms and ranges through comprehensive treatments of direct sums, finiteness conditions, the Wedderburn-Artin Theorem, the Jacobson radical, the hom and tensor functions, Morita equivalence and duality, decomposition theory of injective and projective modules, and semiperfect and perfect rings. In this second edition we have included a chapter containing many of the classical results on artinian rings that have helped to form the foundation for much of the contemporary research on the representation theory of artinian rings and finite dimensional algebras. Both to illustrate the text and to extend it we have included a substantial number of exercises covering a wide spectrum of difficulty. There are, of course, many important areas of ring and module theory that the text does not touch upon. For example, we have made no attempt to cover such subjects as homology, rings of quotients, or commutative ring theory.

This book has evolved from our lectures and research over the past several years. We are deeply indebted to many of our students and colleagues for their ideas and encouragement during its preparation. We extend our sincere thanks to them and to the several people who have helped with the preparation of the manuscripts for the first two editions, and/or pointed out errors in the first.

Finally, we apologize to the many authors whose works we have used but not specifically cited. Virtually all of the results in this book have appeared in some form elsewhere in the literature, and they can be found either in the books and articles that are listed in our bibliography, or in those listed in the collective bibliographies of our citations.

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January 1992

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§0. Preliminaries

In this section is assembled a summary of various bits of notation, terminology, and background information. Of course, we reserve the right to use variations in our notation and terminology that we believe to be self-explanatory without the need of any further comment.

A word about categories. We shall deal only with very special concrete categories and our use of categorical algebra will be really just terminological —at a very elementary level. Here we provide the basic terminology that we shall use and a bit more. We emphasize though that our actual use of it will develop gradually and, we hope, naturally. There is, therefore, no need to try to master it at the beginning.

0.1. Functions. Usually, but not always, we will write functions "on the left". That is, if f is a function from A to B, and if $a \in A$, we write $f(a)$ for the value of f at a. Notation like $f: A \to B$ denotes a function from A to B. The elementwise action of a function $f: A \to B$ is described by

$$f: a \mapsto f(a) \qquad (a \in A).$$

Thus, if $A' \subseteq A$, the *restriction* $(f \mid A')$ of f to A' is defined by

$$(f \mid A'): a' \mapsto f(a') \qquad (a' \in A').$$

Given $f: A \to B$, $A' \subseteq A$, and $B' \subseteq B$, we write

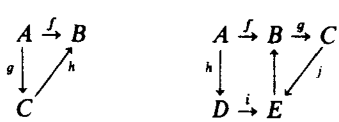
$$f(A') = \{f(a) \mid a \in A'\} \qquad \text{and} \qquad f^{\leftarrow}(B') = \{a \in A \mid f(a) \in B'\}.$$

For the *composite* or *product* of two functions $f: A \to B$ and $g: B \to C$ we write $g \circ f$, or when no ambiguity is threatened, just gf; thus, $g \circ f: A \to C$ is defined by $g \circ f: a \mapsto g(f(a))$ for all $a \in A$. The resulting operation on functions is associative wherever it is defined. The *identity function* from A to itself is denoted by 1_A. The set of all functions from A to B is denoted by B^A or by $Map(A, B)$:

$$B^A = Map(A, B) = \{f \mid f: A \to B\}.$$

So A^A is a monoid (= semigroup with identity) under the operation of composition.

A diagram of sets and functions *commutes* or is *commutative* in case travel around it is independent of path. For example, the first diagram commutes iff $f = hg$. If the second is commutative,



then in particular, travel from A to E is independent of path, whence $jgf = ih$.

A function $f: A \to B$ is *injective* (*surjective*) or is an *injection* (*surjection*) in case it has a *left* (*right*) *inverse* $f': B \to A$; that is, in case $f'f = 1_A$ ($ff' = 1_B$) for some $f': B \to A$. So (see (0.2)) $f: A \to B$ is injective (surjective) iff it is one-to-one (onto B). A function $f: A \to B$ is *bijective* or a *bijection* in case it is both injective and surjective; that is, iff there exists a (necessarily unique) inverse $f^{-1}: B \to A$ with $ff^{-1} = 1_B$ and $f^{-1}f = 1_A$.

If $A \subseteq B$, then the function $i = i_{A \subseteq B}: A \to B$ defined by $i = (1_B | A): a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subseteq B$ and $A \subseteq C$, and if $B \neq C$, then $i_{A \subseteq B} \neq i_{A \subseteq C}$. Of course $1_A = i_{A \subseteq A}$.

With every pair (0, 1) there is a *Kronecker delta*; that is, a function $\delta: (\alpha, \beta) \mapsto \delta_{\alpha\beta}$ on the class of all ordered pairs defined by

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Whenever we use a Kronecker delta, the context will make clear our choice of the pair (0, 1).

0.2. The Axiom of Choice. Let A be a set, let $\mathscr{S}$ be a collection of non-empty subsets of B, and let σ be a function from A to $\mathscr{S}$. Then the *Axiom of Choice* states that there is a function $g: A \to B$ such that

$$g(a) \in \sigma(a) \qquad (a \in A).$$

Suppose now that $f: B \to A$ is onto A; that is, $f(B) = A$. Then for each $a \in A$, there is a non-empty subset $\sigma(a) = f^{\leftarrow}(\{a\}) \subseteq B$. Applying the Axiom of Choice to A, the function $\sigma: a \mapsto \sigma(a)$, and the collection $\mathscr{S}$ of subsets of B produces a right inverse g for f, so as claimed in (0.1), f is surjective.

Let $\sim$ be an equivalence relation on a set A. A subset R of A is a (*complete*) *irredundant set of representatives* of the relation $\sim$ in case for each $a \in A$ there is a unique $\sigma(a) \in R$ such that $a \sim \sigma(a)$. The Axiom of Choice guarantees the existence of such a set of representatives for each equivalence relation.

0.3. Cartesian Products. A function $\sigma: A \to X$ will sometimes be called an *indexed set* (*in X indexed by A*) or an *A-tuple* (in X) and will be written as

$$\sigma = (x_\alpha)_{\alpha \in A}$$

where $x_\alpha = \sigma(\alpha)$. If $A = \{1, \ldots, n\}$, then we also use the standard variation $(x_\alpha)_{\alpha \in A} = (x_1, \ldots, x_n)$. Let $(X_\alpha)_{\alpha \in A}$ be an indexed set of non-empty subsets of a set X. Then the (*cartesian*) *product* of $(X_\alpha)_{\alpha \in A}$ is

$$\mathsf{X}_A X_\alpha = \{\sigma: A \to X \mid \sigma(\alpha) \in X_\alpha \quad (\alpha \in A)\}.$$

That is, $\mathsf{X}_A X_\alpha$ is just the set of all A-tuples $(x_\alpha)_{\alpha \in A}$ such that $x_\alpha \in X_\alpha$ $(\alpha \in A)$. By the Axiom of Choice $\mathsf{X}_A X_\alpha$ is non-empty. If $A = \{1, \ldots, n\}$, then we allow the notational variation

$$\mathsf{X}_A X_\alpha = X_1 \times \ldots \times X_n.$$

Note that if $X = X_\alpha$ $(\alpha \in A)$, then the cartesian product $\mathsf{X}_A X_\alpha$ is simply X^A, the set of all functions from A to X. For each $\alpha \in A$ the *α-projection*

$\pi_\alpha : \mathsf{X}_A X_\alpha \to X_\alpha$ is defined via

$$\pi_\alpha : \sigma \mapsto \sigma(a) \qquad (\sigma \in \mathsf{X}_A X_\alpha).$$

In A-tuple notation, $\pi_\alpha((x_\beta)_{\beta \in A}) = x_\alpha$. An easy application of the Axiom of Choice shows that each π_α is surjective. Observe that if σ and σ' are in this cartesian product, then $\sigma = \sigma'$ iff $\pi_\alpha \sigma = \pi_\alpha \sigma'$ for all $\alpha \in A$. This fact establishes the uniqueness assertion in the following result. This result, whose easy proof we omit, is used in making certain definitions *coordinatewise.*

0.4. *Let $(X_\alpha)_{\alpha \in A}$ be an indexed set of non-empty sets, let Y be a set, and for each $\alpha \in A$, let $f_\alpha : Y \to X_\alpha$. Then there is a unique $f: Y \to \mathsf{X}_A X_\alpha$ such that $\pi_\alpha f = f_\alpha$ for each $\alpha \in A$.*

0.5. Posets and Lattices. A relation $\leq$ on a set P is a *partial order* on P in case it is reflexive ($a \leq a$), transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$), and anti-symmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$). A pair $(P, \leq)$ consisting of a set and a partial order on the set is called a *partially ordered set* or a *poset.* If the partial order is a *total order* ($a \leq b$ or $b \leq a$ for every pair a, b), then the poset is a *chain.* If $(P, \leq)$ is a poset and if $P' \subseteq P$, then $(P', \leq')$ is a subposet in case $\leq'$ is the restriction of $\leq$ to P'; of course, this requires that $(P', \leq')$ be a poset. Henceforth, we will usually identify a poset $(P, \leq)$ with its underlying set P.

Let P be a poset and let $A \subseteq P$. An element $e \in A$ is a *greatest* (*least*) element of A in case $a \leq e$ ($e \leq a$) for all $a \in A$. Not every subset of a poset has a greatest or a least element, but clearly if one does exist, it is unique. (See Example (2) below.) An element $b \in P$ is an *upper bound* (*lower bound*) for A in case $a \leq b$ ($b \leq a$) for all $a \in A$. So a greatest (least) element, if it exists, is an upper (lower) bound for A. If the set of upper bounds of A has a least element, it is called the *least upper bound* (lub), *join,* or *supremum* (sup) of A; if the set of lower bounds has a greatest element, it is called the *greatest lower bound* (glb), *meet,* or *infimum* (inf) of A. A *lattice* (*complete lattice*) is a poset P in which every pair (every subset) of P has both a least upper bound and a greatest lower bound in P.

Examples. (1) Let X be a set. The *power set* of X is the set $\mathscr{P}(X)$ of all subsets of X. Then $\mathscr{P}(X)$ is certainly a poset under the partial order of set inclusion. This poset is a complete lattice for if $\mathscr{A}$ is a subset of $\mathscr{P}(X)$, then its join in $\mathscr{P}(X)$ is its union $\cup\mathscr{A}$ and its meet in $\mathscr{P}(X)$ is its intersection $\cap\mathscr{A}$.

(2) Let X be a set and let $\mathscr{F}(X)$ be the set of all finite subsets of X. Then $\mathscr{F}(X)$ is a poset under set inclusion, and it is a lattice for if $A, B \in \mathscr{F}(X)$, then $A \cup B$ and $A \cap B$ are their join and meet. Since these are also join and meet of A, B in $\mathscr{P}(X)$, it follows that $\mathscr{F}(X)$ is a sublattice of $\mathscr{P}(X)$. But note that if X is infinite, $\mathscr{F}(X)$ is not complete.

(3) Let X be the closed unit interval on the real line. Then the set $\mathscr{I}(X)$ of all closed intervals in X is certainly a subposet of $\mathscr{P}(X)$. Also the intersection (= meet in $\mathscr{P}(X)$) of any subset of $\mathscr{I}(X)$ is again in $\mathscr{I}(X)$. The convex closure of the union of any subset $\mathscr{A}$ of $\mathscr{I}(X)$ is in $\mathscr{I}(X)$ and is clearly the join of $\mathscr{A}$ in $\mathscr{I}(X)$. So $\mathscr{I}(X)$ is a complete lattice. But $\mathscr{I}(X)$ is not a sublattice

of $\mathscr{P}(X)$ precisely because the join in $\mathscr{J}(X)$ of some pairs of elements of $\mathscr{J}(X)$ is not their join (= union) in $\mathscr{P}(X)$.

(4) Let X be a two-dimensional real vector space and let $\mathscr{S}(X)$ be the set of all subspaces. Then $\mathscr{S}(X)$ is a subposet of $\mathscr{P}(X)$, and the intersection of any subset of $\mathscr{S}(X)$ is again in $\mathscr{S}(X)$. The join in $\mathscr{S}(X)$ of any subset $\mathscr{A}$ of $\mathscr{S}(X)$ is the subspace spanned by the union $\cup\mathscr{A}$ (not necessarily $\cup\mathscr{A}$ itself). So $\mathscr{S}(X)$ is a complete lattice but it is not a sublattice of $\mathscr{P}(X)$.

Let P be a lattice. Then each pair $a, b \in P$ has both a join and a meet in P; let us denote these by $a \vee b$ and $a \wedge b$, respectively. Then the maps $\vee$ and $\wedge$ from $P \times P$ to P defined by

$$(a, b) \mapsto a \vee b \qquad \text{and} \qquad (a, b) \mapsto a \wedge b$$

are binary operations on P. It is easy to see that both $(P, \vee)$ and $(P, \wedge)$ are commutative semigroups with

$$a \vee a = a = a \wedge a \qquad (a \in P).$$

The lattice is said to be *modular* in case it satisfies the *modularity condition*: For all $a, b, c \in P$

$$a \geq b \text{ implies } a \wedge (b \vee c) = b \vee (a \wedge c).$$

Most lattices we encounter will be modular (but note (3) above). The lattice is *distributive* in case it satisfies the stronger property: For all $a, b, c \in P$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Examples (1) and (2) above are distributive, but (4) is not.

0.6. *A partially ordered set P is a complete lattice if P has a join (i.e., P contains a greatest element) and every non-empty subset of P has a meet in P.*

Proof. It will suffice to prove that if $B \subseteq P$, then B has a join in P. Let $e \in P$ be the greatest element of P. Then $e \geq x$ for all $x \in P$. In particular, the set of upper bounds of B is non-empty, so it has a meet. Clearly this meet of the upper bounds of B is an upper bound of B and hence the join of B. □

0.7. Lattice Homomorphisms. Let P and P' be posets. A map $f: P \to P'$ is *order preserving* (*order reversing*) in case whenever $a \leq b$ in P, then $f(a) \leq f(b)$ $(f(b) \leq f(a))$ in P'. If P and P' are lattices, then f is a *lattice homomorphism* (*lattice antihomomorphism*) in case whenever $a, b \in P$,

$$f(a \vee b) = f(a) \vee f(b) \qquad (f(a \vee b) = f(a) \wedge f(b))$$
$$f(a \wedge b) = f(a) \wedge f(b) \qquad (f(a \wedge b) = f(a) \vee f(b)).$$

It is easy to see (using $a \leq b \Leftrightarrow a = a \wedge b$) that a lattice-homomorphism is order preserving. The converse, however, is false (try the inclusion map $\mathscr{J}(X) \to \mathscr{P}(X)$ in example (3) of (0.5)). A bijective lattice (anti-) homomorphism is a *lattice* (*anti-*)*isomorphism*. It is a simple exercise to prove the following useful test:

0.8. *Let P and P' be lattices, and let $f: P \to P'$ be bijective with inverse $f^{-1}: P' \to P$. Then f is a lattice isomorphism if and only if both f and f^{-1} are order preserving.*

0.9. The Maximal Principle. Let P be a poset. An element $m \in P$ is *maximal* (*minimal*) in P in case $x \in P$ and $x \geq m$ ($x \leq m$) implies $x = m$. Clearly, a greatest (least) element in P, if it exists, is maximal (minimal) in P; on the other hand, a poset may have many maximal (minimal) elements and no greatest (least) element.

A poset P is *inductive* in case every subchain of P has an upper bound in P; that is, for every subset C of P that is totally ordered by the partial ordering of P, there is an element of P greater than or equal to every element of C. *The Maximal Principle* (frequently called Zorn's Lemma) is an equivalent form of the Axiom of Choice (see Stoll [63] for the details). It states:

Every non-empty inductive poset has at least one maximal element.

0.10. Cardinal Numbers. Two sets A and B are *cardinally equivalent* or have the *same cardinal* in case there is a bijection from A to B (and hence one from B to A). Since this clearly defines an equivalence relation, the class of all sets (see (0.11)) can be partitioned into its classes of cardinally equivalent sets. These classes are the *cardinal numbers*. The class of a set A is denoted by *card* A:

$$\textit{card}\, A = \{B \mid \text{there is a bijection } A \to B\}.$$

Given two sets A and B we write

$$\textit{card}\, A \leq \textit{card}\, B$$

in case there is an injection from A to B (or, equivalently, a surjection from B to A). Clearly this is independent of the representatives A and B. Given sets A and B there is always an injection from one to the other. The Cantor–Schröder–Bernstein Theorem states that

If $\textit{card}\, A \leq \textit{card}\, B$ *and* $\textit{card}\, B \leq \textit{card}\, A$, *then* $\textit{card}\, A = \textit{card}\, B$.

Thus the relation $\leq$ is a total order on the class of cardinal numbers.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the natural numbers. Its cardinality is often denoted by $\textit{card}\, \mathbb{N} = \aleph_0$. A set A is *finite* if $\textit{card}\, A < \textit{card}\, \mathbb{N}$. Of course, $\textit{card}\,(\{1, \ldots, n\}) = n$ and card $\varnothing = 0$. If $\textit{card}\, A \leq \textit{card}\, \mathbb{N}$, then A is *countable*. If $\textit{card}\, A \geq \textit{card}\, \mathbb{N}$, then A is *infinite*.

The operations of cardinal arithmetic are given by

$$\textit{card}\, A + \textit{card}\, B = \textit{card}((A \times \{1\}) \cup (B \times \{2\}))$$

$$\textit{card}\, A \cdot \textit{card}\, B = \textit{card}(A \times B)$$

$$(\textit{card}\, A)^{(\textit{card}\, B)} = \textit{card}(A^B)$$

If A and B are finite sets these operations agree with ordinary addition, multiplication and exponentiation. Moreover, they satisfy:

(1) *If A is infinite then, $card\ A + card\ B = \max\{card\ A, card\ B\}$.*
(2) *If A is infinite and $B \neq \varnothing$, then*

$$card\ A \cdot card\ B = \max\{card\ A, card\ B\}.$$

(3) *For all sets A, B, and C,*

$$((card\ A)^{(card\ B)})^{(card\ C)} = (card\ A)^{(card\ B)\cdot(card\ C)}.$$

(4) *If $card\ B \geq 2$, then $(card\ B)^{(card\ A)} > card\ A$.*

It is easy to establish the existence of a bijection between the power set $\mathscr{P}(A)$ and the set of functions from A to $\{1, 2\}$. Thus $card(\mathscr{P}(A)) = 2^{(card\ A)} > card\ A$. However, the set of finite subsets of any infinite set A has the same cardinality as A. For further details see Stoll [63].

0.11. Categories. The term "class", like that of "set", will be undefined. Every set is a class, and there is a class containing all sets. Note that if A is a set and $\mathscr{C}$ is a class, then an indexed class $(A_C)_{C \in \mathscr{C}}$ in $\mathscr{P}(A)$ has a union and an intersection in A. Let $\mathscr{C}$ be a class for each pair $A, B \in \mathscr{C}$, let $mor_{\mathsf{C}}(A, B)$ be a set; write the elements of $mor_{\mathsf{C}}(A, B)$ as "arrows" $f: A \to B$ for which A is called the *domain* and B the *codomain*. Finally, suppose that for each triple $A, B, C \in \mathscr{C}$ there is a function

$$\circ: mor_{\mathsf{C}}(B, C) \times mor_{\mathsf{C}}(A, B) \to mor_{\mathsf{C}}(A, C).$$

We denote the arrow assigned to a pair

$$g: B \to C \qquad f: A \to B$$

by the arrow $gf: A \to C$. The system $\mathsf{C} = (\mathscr{C}, mor_{\mathsf{C}}, \circ)$ consisting of the class $\mathscr{C}$, the map $mor_{\mathsf{C}}: (A, B) \mapsto mor_{\mathsf{C}}(A, B)$, and the rule $\circ$ is a *category* in case:

(C.1) For every triple $h: C \to D$, $g: B \to C$, $f: A \to B$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(C.2) For each $A \in \mathscr{C}$, there is a unique $1_A \in mor_{\mathsf{C}}(A, A)$ such that if $f: A \to B$ and $g: C \to A$, then

$$f \circ 1_A = f \qquad \text{and} \qquad 1_A \circ g = g.$$

If C is a category, then the elements of the class $\mathscr{C}$ are called the *objects* of the category, the "arrows" $f: A \to B$ are called the *morphisms*, the partial map $\circ$ is called the *composition*, and the morphisms 1_A are called the *identities* of the category. A morphism $f: A \to B$ in C is called an *isomorphism* in case there is a (necessarily unique) morphism $f^{-1}: B \to A$ in C such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

For our purpose the most interesting categories are certain "concrete" categories. Let $\mathsf{C} = (\mathscr{C}, mor_{\mathsf{C}}, \circ)$ be a category. Then C is *concrete* in case there is a function u from $\mathscr{C}$ to the class of sets such that for each $A, B \in \mathscr{C}$

$$mor_{\mathsf{C}}(A, B) \subseteq Map(u(A), u(B)),$$

$$1_A = 1_{u(A)},$$

and such that $\circ$ is the usual composition of functions. Here an isomorphism $f:A \to B$ is a bijection $f:u(A) \to u(B)$.

Examples. (1) Let $\mathscr{S}$ be the class of all sets; for each $A, B \in \mathscr{S}$, let $mor_{\mathsf{S}}(A, B) = Map(A, B)$, and for each $A, B, C \in \mathscr{S}$, let $\circ: mor_{\mathsf{S}}(B, C) \times mor_{\mathsf{S}}(A, B) \to mor_{\mathsf{S}}(A, C)$ be the composition of functions. Then $\mathsf{S} = (\mathscr{S}, mor_{\mathsf{S}}, \circ)$ is a concrete category where $u(A) = A$ for each $A \in \mathscr{S}$. Call **S** *the category of sets.*

(2) Let $\mathscr{G}$ be the class of all groups, let $mor_{\mathsf{G}}(G, H)$ be the set of all group homomorphisms from G to H, and again let $\circ$ be the usual composition of functions. Then $\mathsf{G} = (\mathscr{G}, mor_{\mathsf{G}}, \circ)$ is a concrete category, the *category of groups*, where $u(G)$ is the underlying set of G.

(3) The *category of* **V** *real vector* spaces is the category $(\mathscr{V}, mor_{\mathsf{V}}, \circ)$ where $\mathscr{V}$ is the class of real vector spaces, $mor_{\mathsf{V}}(U, V)$ is the set of linear transformations from U to V, and $\circ$ is the usual composition. This category is concrete where $u(V)$ is the underlying set of V.

(4) Let $\mathscr{P}$ be the class of all posets, $mor_{\mathsf{P}}(P, Q)$ the set of all monotone maps (order preserving and order reversing ones), and $\circ$ the usual composition. Then $(\mathscr{P}, mor_{\mathsf{P}}, \circ)$ is not a category, for $\circ$ is not as required—the composite of two monotone functions need not be monotone.

If $\mathsf{C} = (\mathscr{C}, mor_{\mathsf{C}}, \circ)$ is a concrete category, then the set $u(A)$ is called the *underlying* set of $A \in \mathscr{C}$.

A category $\mathsf{D} = (\mathscr{D}, mor_{\mathsf{D}}, \circ)$ is a *subcategory* of $\mathsf{C} = (\mathscr{C}, mor_{\mathsf{C}}, \circ)$ provided $\mathscr{D} \subseteq \mathscr{C}$, $mor_{\mathsf{D}}(A, B) \subseteq mor_{\mathsf{C}}(A, B)$ for each pair $A, B \in \mathscr{D}$, $\circ$ in **D** is the restriction of $\circ$ in **C**. If in addition $mor_{\mathsf{D}}(A, B) = mor_{\mathsf{C}}(A, B)$ for each $A, B \in \mathscr{D}$, then **D** is a *full* subcategory of **C**.

It is clear that the class of abelian groups is the class of objects of a full subcategory of the category of groups, and that this category has a full subcategory whose objects are the finite abelian groups. It is a common practice in algebra to identify an object in a category with its underlying set. Thus for example, we usually identify a group $(G, \circ)$, consisting of a set G and an operation $\circ$, with its underlying set G. Note, however, that the category of groups is not a subcategory of the category of sets, quite simply because for groups $(G, \circ)$, $(H, \circ)$ in $\mathscr{G}$

$$mor_{\mathsf{G}}((G, \circ), (H, \circ)) \subseteq Map(G, H)$$

and

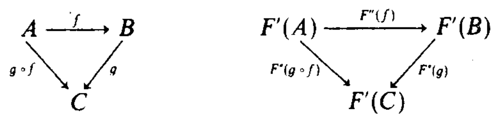
$$mor_{\mathsf{G}}((G, \circ), (H, \circ)) \nsubseteq Map((G, \circ), (H, \circ)).$$

0.12. Functors. A functor is a thing that can be viewed as a "homomorphism of categories". Let $\mathsf{C} = (\mathscr{C}, mor_{\mathsf{C}}, \circ)$ and $\mathsf{D} = (\mathscr{D}, map_{\mathsf{D}}, \circ)$ be two categories. A pair of functions $F = (F', F'')$ is a *covariant functor* from **C** to **D** in case F' is a function from $\mathscr{C}$ to $\mathscr{D}$, F'' is a function from the morphisms of **C** to those of **D** such that for all $A, B, C \in \mathscr{C}$ and all $f:A \to B$ and $g:B \to C$ in **C**,

(F.1) $F''(f):F'(A) \to F'(B)$ in **D**;

(F.2) $F''(g \circ f) = F''(g) \circ F''(f)$;
(F.3) $F''(1_A) = 1_{F'(A)}$.

Thus, a covariant functor sends objects to objects, maps to maps, identities to identities, and "preserves commuting triangles":



A *contravariant functor* is a pair $F = (F', F'')$ satisfying instead of (F.1) and (F.2) their duals

(F.1)* $F''(f): F'(B) \to F'(A)$ in $\mathbf{D}$;
(F.2)* $F''(g \circ f) = F''(f) \circ F''(g)$;
(F.3) $F''(1_A) = 1_{F'(A)}$.

So a contravariant functor is "arrow reversing".

Examples. (1) Given a category $\mathbf{C} = (\mathscr{C}, mor_{\mathbf{C}}, \circ)$, there is the *identity functor* $1_{\mathbf{C}} = (1'_{\mathbf{C}}, 1''_{\mathbf{C}})$ from $\mathbf{C}$ to $\mathbf{C}$ defined by $1'_{\mathbf{C}}(A) = A$ and $1''_{\mathbf{C}}(f) = f$.

(2) Let $\mathbf{C} = (\mathscr{C}, mor_{\mathbf{C}}, \circ)$ be a concrete category. For each $A \in \mathscr{C}$, let $F'(A) = u(A)$ be the underlying set of A. For each morphism f of $\mathbf{C}$, let $F''(f) = f$. Then clearly $F = (F', F'')$ is a covariant functor from $\mathbf{C}$ to the category of sets. It is called a *forgetful functor* (because it "forgets" all the "structure" on the objects of $\mathbf{C}$). It should be evident there are "partially forgetful functors" of various kinds—for example, the covariant functor from the category of real vector spaces to the category of abelian groups that "forgets" the scalar multiplication.

(3) Let $(G, +)$ be an abelian group. If A is a set, then $(G^A, +)$ is an abelian group where for $\sigma, \tau \in G^A$, the sum $\sigma + \tau \in G^A$ is defined by $(\sigma + \tau): a \mapsto \sigma(a) + \tau(a)$. (Note that $(G^A, +)$ is simply the cartesian product of A copies of G with coordinatewise addition.) Define $F'(A) = (G^A, +)$. If A, B are sets, and if $f: A \to B$, then define $F''(f): G^B \to G^A$ by

$$F''(f)(\sigma) = \sigma \circ f \qquad (\sigma \in G^B).$$

Then $F''(f)$ is a group homomorphism, and $F = (F', F'')$ is a contravariant functor from the category of non-empty sets to the category of abelian groups. All kinds of contravariant functors can be built in this way. For example, if $(G, +, \circ)$ were a real vector space, then G^A can be made into a vector space with coordinatewise operations, and a contravariant functor into the real vector spaces results.

Given a functor $F = (F', F'')$, then rather than bother with all the primes, we shall usually write $F(A)$ and $F(f)$ instead of $F'(A)$ and $F''(f)$. The relatively minor formal objection is that a morphism f of the category may also be an object of the category whence $F'(f)$ and $F''(f)$ may both make sense yet be different.

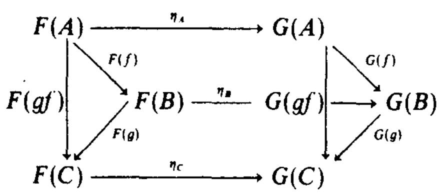
0.13. Natural Transformations. A natural transformation is a thing that compares two functors between the same categories. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Let F and G be functors from C to D, say both covariant. Let $\eta = (\eta_A)_{A \in C}$ be an indexed class of morphisms in D indexed by $\mathscr{C}$ such that for each $A \in \mathscr{C}$,

$$\eta_A \in mor_{\mathsf{D}}(F(A), G(A)).$$

Then η is a *natural transformation* from F to G in case for each pair, $A, B \in \mathscr{C}$, and each $f \in mor_{\mathsf{C}}(A, B)$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes; that is $\eta_B \circ F(f) = G(f) \circ \eta_A$. If each η_A is an isomorphism, then η is called a *natural isomorphism*. (If both F and G were contravariant, the only change would be to reverse the arrows $F(f)$ and $G(f)$.) The crucial property of functors is that "they preserve commuting triangles"; then a natural transformation η achieves a "translation of commuting triangles"



In fact notice that any commutative diagram Δ in C when operated on elementwise by F and G produces a pair of commutative diagrams $F(\Delta)$ and $G(\Delta)$ in D (because F and G are functors). Then a natural transformation η from F to G "translates" commutatively $F(\Delta)$ onto $G(\Delta)$. Because of the technical clumsiness in defining many interesting functors at this stage, we shall postpone giving examples until such time as we have an abundance of functors (see §20).

Some Special Notation

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the non negative integers;
$\mathbb{N} = \{1, 2, \ldots\}$, the positive integers;
$\mathbb{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$;
$\mathbb{Z}$ = the set of integers;
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$;
$\mathbb{Q}$ = the set of rational numbers;
$\mathbb{R}$ = the set of real numbers;
$\mathbb{C}$ = the set of complex numbers;
$\varnothing$ = the empty set.

Chapter 1

Rings, Modules and Homomorphisms

The subject of our study is ring theory. In this chapter we introduce the fundamental tools of this study. Section 1 reviews the basic facts about rings, subrings, ideals, and ring homomorphisms. It also introduces some of the notation and the examples that will be needed later.

Rings admit a valuable and natural representation theory, analogous to the permutation representation theory for groups. As we shall see, each ring admits a vast horde of representations as an endomorphism ring of an abelian group. Each of these representations is called a *module.* A substantial amount of information about a ring can be learned from a study of the class of modules it admits. Modules actually serve as a generalization of both vector spaces and abelian groups, and their basic behavior is quite similar to that of the more special systems. In Sections 2 and 3 we introduce modules and their homomorphisms. In Section 4 we see that these form various natural and important categories, and we begin our study of categories of modules.

§1. Review of Rings and their Homomorphisms

Rings and Subrings

By a ring we shall always mean an associative ring with identity. Formally, then, a *ring* is a system $(R, +, \cdot, 0, 1)$ consisting of a set R, two binary operations, addition $(+)$ and multiplication $(\cdot)$, and two elements $0 \neq 1$ of R such that $(R, +, 0)$ is an abelian group, $(R, \cdot, 1)$ is a monoid (i.e., a semigroup with identity 1) and multiplication is both left and right distributive over addition. A ring whose multiplicative structure is commutative is called a *commutative ring.* We assume that the reader is versed in the elementary arithmetic of rings and we shall therefore use that arithmetic without further mention. We shall also invoke the time-honored convention of identifying a ring with its underlying set whenever there is no real risk of confusion. Of course, when we are dealing with more than one ring we may modify our notation to eliminate ambiguity. Thus, for example, if R and S are two rings, we may distinguish their identities by such self-explanatory notation as 1_R and 1_S.

Often in practice, particularly in some areas of analysis, one encounters "rings without identity". Nevertheless the severity of our requirement of an identity is more imaginary than real. Indeed a ring without identity can be