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（英文版·第3版）

COMPLEX ANALYSIS

Third Edition

Lars V. Ahlfors

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Mathematics Series

（美）Lars V. Ahlfors 著

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Complex Analysis

(Third Edition)

本书的诞生还是半个世纪之前的事情，但是，深贯其中的严谨的学术风范以及针对不同时代所做出的切实改进使得它愈久弥新，成为复分析领域历经考验的一本经典教材。本书作者在数学分析领域声名卓著，多次荣获国际大奖，这也是本书始终保持旺盛的生命力的原因之一。

本书适合用做数学专业本科高年级学生及研究生教材。

作者简介

Lars V. Ahlfors 生前是哈佛大学数学教授。他于1924年进入赫尔辛基大学学习，并在1930年于芬兰著名的土尔库大学获得博士学位。期间他还师从著名数学家Nevanlinna共同进行研究工作。1936年荣获菲尔茨奖。第二次世界大战结束后，辗转到哈佛大学从事教学工作。他又于1968年和1981年分别荣获Vihuri奖和Wolf奖。他的著述很多，除本书外，还著有《Riemann Surfaces》和《Conformal Invariants》等。

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Preface

Complex Analysis has successfully maintained its place as the standard elementary text on functions of one complex variable. There is, nevertheless, need for a new edition, partly because of changes in current mathematical terminology, partly because of differences in student preparedness and aims.

There are no radical innovations in the new edition. The author still believes strongly in a geometric approach to the basics, and for this reason the introductory chapters are virtually unchanged. In a few places, throughout the book, it was desirable to clarify certain points that experience has shown to have been a source of possible misunderstanding or difficulties. Misprints and minor errors that have come to my attention have been corrected. Otherwise, the main differences between the second and third editions can be summarized as follows:

1. Notations and terminology have been modernized, but it did not seem necessary to change the style in any significant way.

2. In Chapter 2 a brief section on the change of length and area under conformal mapping has been added. To some degree this infringes on the otherwise self-contained exposition, for it forces the reader to fall back on calculus for the definition and manipulation of double integrals. The disadvantage is minor.

3. In Chapter 4 there is a new and simpler proof of the general form of Cauchy's theorem. It is due to A. F. Beardon, who has kindly permitted me to reproduce it. It complements but does not replace the old proof, which has been retained and improved.

4. A short section on the Riemann zeta function has been included.

This always fascinates students, and the proof of the functional equation illustrates the use of residues in a less trivial situation than the mere computation of definite integrals.

5. Large parts of Chapter 8 have been completely rewritten. The main purpose was to introduce the reader to the terminology of germs and sheaves while emphasizing all the classical concepts. It goes without saying that nothing beyond the basic notions of sheaf theory would have been compatible with the elementary nature of the book.

6. The author has successfully resisted the temptation to include Riemann surfaces as one-dimensional complex manifolds. The book would lose much of its usefulness if it went beyond its purpose of being no more than an introduction to the basic methods and results of complex function theory in the plane.

It is my pleasant duty to thank the many who have helped me by pointing out misprints, weaknesses, and errors in the second edition. I am particularly grateful to my colleague Lynn Loomis, who kindly let me share student reaction to a recent course based on my book.

Lars V. Ahlfors

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1 COMPLEX NUMBERS

1. THE ALGEBRA OF COMPLEX NUMBERS

It is fundamental that real and complex numbers obey the same basic laws of arithmetic. We begin our study of complex function theory by stressing and implementing this analogy.

1.1. Arithmetic Operations. From elementary algebra the reader is acquainted with the *imaginary unit* i with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a *complex number* $\alpha + i\beta$. α and β are the *real* and *imaginary part* of the complex number. If $\alpha = 0$, the number is said to be *purely imaginary;* if $\beta = 0$, it is of course *real.* Zero is the only number which is at once real and purely imaginary. Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

Addition and multiplication do not lead out from the system of complex numbers. Assuming that the ordinary rules of arithmetic apply to complex numbers we find indeed

$$(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta) \tag{1}$$

and

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma). \tag{2}$$

In the second identity we have made use of the relation $i^2 = -1$.

It is less obvious that division is also possible. We wish to show that $(\alpha + i\beta)/(\gamma + i\delta)$ is a complex number, provided that $\gamma + i\delta \neq 0$. If the quotient is denoted by $x + iy$, we must have

$$\alpha + i\beta = (\gamma + i\delta)(x + iy).$$

By (2) this condition can be written

$$\alpha + i\beta = (\gamma x - \delta y) + i(\delta x + \gamma y),$$

and we obtain the two equations

$$\begin{aligned} \alpha &= \gamma x - \delta y \\ \beta &= \delta x + \gamma y. \end{aligned}$$

This system of simultaneous linear equations has the unique solution

$$x = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2}$$

$$y = \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2},$$

for we know that $\gamma^2 + \delta^2$ is not zero. We have thus the result

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i\,\frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}. \tag{3}$$

Once the existence of the quotient has been proved, its value can be found in a simpler way. If numerator and denominator are multiplied with $\gamma - i\delta$, we find at once

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{(\alpha + i\beta)(\gamma - i\delta)}{(\gamma + i\delta)(\gamma - i\delta)} = \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2}.$$

As a special case the reciprocal of a complex number $\neq 0$ is given by

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}.$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0, 1, 2, 3.

EXERCISES

1. Find the values of

$$(1 + 2i)^3, \qquad \frac{5}{-3 + 4i}, \qquad \left(\frac{2 + i}{3 - 2i}\right)^2, \qquad (1 + i)^n + (1 - i)^n.$$

2. If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}.$$

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

1.2. Square Roots. We shall now show that the square root of a complex number can be found explicitly. If the given number is $\alpha + i\beta$, we are looking for a number $x + iy$ such that

$$(x + iy)^2 = \alpha + i\beta.$$

This is equivalent to the system of equations

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta. \end{aligned} \tag{4}$$

From these equations we obtain

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2.$$

Hence we must have

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2},$$

where the square root is positive or zero. Together with the first equation (4) we find

$$\begin{aligned} x^2 &= \tfrac{1}{2}(\alpha + \sqrt{\alpha^2 + \beta^2}) \\ y^2 &= \tfrac{1}{2}(-\alpha + \sqrt{\alpha^2 + \beta^2}). \end{aligned} \tag{5}$$

Observe that these quantities are positive or zero regardless of the sign of α.

The equations (5) yield, in general, two opposite values for x and two for y. But these values cannot be combined arbitrarily, for the second equation (4) is not a consequence of (5). We must therefore be careful to select x and y so that their product has the sign of β. This leads to the general solution

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right) \tag{6}$$

provided that $\beta \neq 0$. For $\beta = 0$ the values are $\pm\sqrt{\alpha}$ if $\alpha \geqq 0$, $\pm i\sqrt{-\alpha}$

if $\alpha < 0$. It is understood that all square roots of positive numbers are taken with the positive sign.

We have found that the square root of any complex number exists and has two opposite values. They coincide only if $\alpha + i\beta = 0$. They are real if $\beta = 0$, $\alpha \geqq 0$ and purely imaginary if $\beta = 0$, $\alpha \leqq 0$. In other words, except for zero, only positive numbers have real square roots and only negative numbers have purely imaginary square roots.

Since both square roots are in general complex, it is not possible to distinguish between the positive and negative square root of a complex number. We could of course distinguish between the upper and lower sign in (6), but this distinction is artificial and should be avoided. The correct way is to treat both square roots in a symmetric manner.

EXERCISES

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$$

2. Find the four values of $\sqrt[4]{-1}$.

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

1.3. Justification. So far our approach to complex numbers has been completely uncritical. We have not questioned the existence of a number system in which the equation $x^2 + 1 = 0$ has a solution while all the rules of arithmetic remain in force.

We begin by recalling the characteristic properties of the real-number system which we denote by **R**. In the first place, **R** is a *field*. This means that addition and multiplication are defined, satisfying the *associative, commutative,* and *distributive laws*. The numbers 0 and 1 are neutral elements under addition and multiplication, respectively: $\alpha + 0 = \alpha$, $\alpha \cdot 1 = \alpha$ for all α. Moreover, the equation of subtraction $\beta + x = \alpha$ has always a solution, and the equation of division $\beta x = \alpha$ has a solution whenever $\beta \neq 0$.†

One shows by elementary reasoning that the neutral elements and the results of subtraction and division are unique. Also, every field is an *integral domain:* $\alpha\beta = 0$ if and only if $\alpha = 0$ or $\beta = 0$.

† We assume that the reader has a working knowledge of elementary algebra. Although the above characterization of a field is complete, it obviously does not convey much to a student who is not already at least vaguely familiar with the concept.

These properties are common to all fields. In addition, the field **R** has an *order relation* $\alpha < \beta$ (or $\beta > \alpha$). It is most easily defined in terms of the set $\mathbf{R}^+$ of *positive* real numbers: $\alpha < \beta$ if and only if $\beta - \alpha \in \mathbf{R}^+$. The set $\mathbf{R}^+$ is characterized by the following properties: (1) 0 is not a positive number; (2) if $\alpha \neq 0$ either α or $-\alpha$ is positive; (3) the sum and the product of two positive numbers are positive. From these conditions one derives all the usual rules for manipulation of inequalities. In particular one finds that every square α^2 is either positive or zero; therefore $1 = 1^2$ is a positive number.

By virtue of the order relation the sums $1, 1 + 1, 1 + 1 + 1, \ldots$ are all different. Hence **R** contains the natural numbers, and since it is a field it must contain the subfield formed by all rational numbers.

Finally, **R** satisfies the following *completeness condition:* every increasing and bounded sequence of real numbers has a limit. Let $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \cdots$, and assume the existence of a real number B such that $\alpha_n < B$ for all n. Then the completeness condition requires the existence of a number $A = \lim_{n\to\infty} \alpha_n$ with the following property: given any $\varepsilon > 0$ there exists a natural number n_0 such that $A - \varepsilon < \alpha_n < A$ for all $n > n_0$.

Our discussion of the real-number system is incomplete inasmuch as we have not proved the existence and uniqueness (up to isomorphisms) of a system **R** with the postulated properties.† The student who is not thoroughly familiar with one of the constructive processes by which real numbers can be introduced should not fail to fill this gap by consulting any textbook in which a full axiomatic treatment of real numbers is given.

The equation $x^2 + 1 = 0$ has no solution in **R**, for $\alpha^2 + 1$ is always positive. Suppose now that a field **F** can be found which contains **R** as a subfield, and in which the equation $x^2 + 1 = 0$ can be solved. Denote a solution by i. Then $x^2 + 1 = (x + i)(x - i)$, and the equation $x^2 + 1 = 0$ has exactly two roots in **F**, i and $-i$. Let **C** be the subset of **F** consisting of all elements which can be expressed in the form $\alpha + i\beta$ with real α and β. This representation is unique, for $\alpha + i\beta = \alpha' + i\beta'$ implies $\alpha - \alpha' = -i(\beta - \beta')$; hence $(\alpha - \alpha')^2 = -(\beta - \beta')^2$, and this is possible only if $\alpha = \alpha'$, $\beta = \beta'$.

The subset **C** is a subfield of **F**. In fact, except for trivial verifications which the reader is asked to carry out, this is exactly what was shown in Sec. 1.1. What is more, the structure of **C** is independent of **F**. For if **F**′ is another field containing **R** and a root i' of the equation $x^2 + 1 = 0$,

† An *isomorphism* between two fields is a one-to-one correspondence which preserves sums and products. The word is used quite generally to indicate a correspondence which is one to one and preserves all relations that are considered important in a given connection.

the corresponding subset $\mathbf{C}'$ is formed by all elements $\alpha + i'\beta$. There is a one-to-one correspondence between $\mathbf{C}$ and $\mathbf{C}'$ which associates $\alpha + i\beta$ and $\alpha + i'\beta$, and this correspondence is evidently a field isomorphism. It is thus demonstrated that $\mathbf{C}$ and $\mathbf{C}'$ are isomorphic.

We now define the field of *complex numbers* to be the subfield $\mathbf{C}$ of an arbitrarily given $\mathbf{F}$. We have just seen that the choice of $\mathbf{F}$ makes no difference, but we have not yet shown that there exists a field $\mathbf{F}$ with the required properties. In order to give our definition a meaning it remains to exhibit a field $\mathbf{F}$ which contains $\mathbf{R}$ (or a subfield isomorphic with $\mathbf{R}$) and in which the equation $x^2 + 1 = 0$ has a root.

There are many ways in which such a field can be constructed. The simplest and most direct method is the following: Consider all expressions of the form $\alpha + i\beta$ where α, β are real numbers while the signs $+$ and i are pure symbols ($+$ does *not* indicate addition, and i is *not* an element of a field). These expressions are elements of a field $\mathbf{F}$ in which addition and multiplication are defined by (1) and (2) (observe the two different meanings of the sign $+$). The elements of the particular form $\alpha + i0$ are seen to constitute a subfield isomorphic to $\mathbf{R}$, and the element $0 + i1$ satisfies the equation $x^2 + 1 = 0$; we obtain in fact $(0 + i1)^2 = -(1 + i0)$. The field $\mathbf{F}$ has thus the required properties; moreover, it is identical with the corresponding subfield $\mathbf{C}$, for we can write

$$\alpha + i\beta = (\alpha + i0) + \beta(0 + i1).$$

The existence of the complex-number field is now proved, and we can go back to the simpler notation $\alpha + i\beta$ where the $+$ indicates addition in $\mathbf{C}$ and i is a root of the equation $x^2 + 1 = 0$.

EXERCISES (For students with a background in algebra)

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex-number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.4. Conjugation, Absolute Value. A complex number can be denoted either by a single letter a, representing an element of the field $\mathbf{C}$, or in the form $\alpha + i\beta$ with real α and β. Other standard notations are $z = x + iy$, $\zeta = \xi + i\eta$, $w = u + iv$, and when used in this connection it