SOME CLASSES OF PARTIAL DIFFERENTIAL EQUATIONS

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Translated from the Russian by
H. Zahavi

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PREFACE

This book analyses problems of the theory of partial differential equations which have concerned me over the last few decades. It discusses second-order equations with degeneracy of type and order, equations of mixed elliptic – hyperbolic type, and second-order systems with non-split principal parts, for which classical problems generally cease to be correctly formulated. A separate chapter examines the structural and qualitative properties of classes of nonlinear equations, encompassing some versions of equations of the gravitational field, the theory of waves in a liquid of variable density, etc.

The book has an introductory chapter containing a brief review of the results and methods of the classical theory of linear partial differential equations. This chapter is intended to help the reader understand the complexity and importance of problems which do not satisfy the standard conditions of their normal solvability. A relatively small part of the book caused, to a considerable extent, a rejection of the generalizing conclusions when discussing the material presented in it. For this reason, the results of a number of mathematicians – even when occasionally extremely interesting – have not been included.

I would be pleased if the book were in some way useful to the reader. I would like to thank E. G. Evseev, D. V. Izyumova and S. S. Kharibegashvili for their help in preparing the manuscipt.

A. V. Bitsadze

Moscow-Tbilisi,

CONTENTS

Chapter 1. INTRODUCTION

§ 1. THE CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

1. The concept of partial differential equations. Suppose $F(x, \ldots, p_{i_1 \ldots i_n}, \ldots)$ is a specified N-dimensional vector, $N \geq 1$, the components of which $F_1, \ldots, F_N$ are functions of the points x of the domain D of the space E_n of the independent variables $x_1, \ldots, x_n$, $n > 1$, and *the* M-dimensional vectors

$$p_{i_1 \ldots i_n} = (p^1_{i_1 \ldots i_n}, \ldots, p^M_{i_1 \ldots i_n})$$

with the nonnegative integral indices $i_1, \ldots, i_n$:

$$\sum_{j=1}^{n} i_j = k, \quad k = 0, \ldots, m, \quad m \geq 1.$$

An equation of the form

$$F\left(x, \ldots, \frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \ldots\right) = 0 \tag{1.1}$$

is called a partial differential equation with respect to the unknown vector u with the components $u_1, \ldots, u_M$.

When $N = M = 1$, Eq. (1.1) is a (scalar) equation, and when $N > 1$ it is a system of partial differential equations. The highest order of the derivatives of the required functions which occur in a given equation of system (1.1) is called the order of this equation. The number of equations N and the number of unknown functions M in system (1.1) generally may differ.

The vector $u = (u_1, \ldots, u_M)$, defined in the domain D, which has classical or generalized partial differentials that occur on the left-hand side of Eq. (1.1), and which converts* it to an identity, is called the solution of this equation.

If the dependence of F on all $\frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$ is linear, Eq. (1.1) is called a linear equation.

2. Division by type of partial differential equation. Without loss of generality we can assume that the quantities which occur in Eq. (1.1) are all real. We will assume that $N = M$ and the order of each of the equations occurring in (1.1) equals m.

When the partial differential functions $F_1, \ldots, F_N$ are continuous with respect to all $p_{i_1 \ldots i_n}$, $\sum_{j=1}^{n} i_j = m$, the linear parts of the increments of these functions

$$\sum_{i_1 \ldots i_n} \sum_{j=1}^{N} \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} dp^j_{i_1 \ldots i_n}, \quad i = 1, \ldots, N,$$

are the principal parts. It is therefore natural that they can play a specific role when we examine Eq. (1.1).

Using the square matrices

$$\left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\|, \quad i, j = 1, \ldots, N,$$

we shall construct the Nm-th order form with respect to the real parameters $\lambda_1, \ldots, \lambda_n$:

$$K(\lambda_1, \ldots, \lambda_n) = \det \sum_{i_1 \ldots i_n} \left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\| \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \qquad (1.2)$$

where the sum is taken over all possible nonnegative integer-values of the indices $i_1, \ldots, i_n$, satisfying the condition $\sum_{j=1}^{n} i_j = m$.

Eq. (1.2), in which

$$p^j_{i_1 \ldots i_n} = \frac{\partial^m u_j}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \quad j = 1, \ldots, N,$$

is called the characteristic form (characteristic determinant) of Eq. (1.1). In the linear case the coefficients of this form depend only on the point x, and in the general case they are also functions of the required solution u and its derivatives.

Partial differential equations are divided by type according to the form of (1.2). We will first assume that Eq. (1.1) is linear. If, for the fixed point $x \in D$, we can obtain the affine transformation of the variables

$$\lambda_i = \lambda_i(\mu_1, \ldots, \mu_n), \quad i = 1, \ldots, n,$$

as a result of which the form obtained from (1.2) contains only l, $0 < l < n$, variables μ_i, we can then say that Eq. (1.1) parabolically degenerates at the point x. Assuming that there is no parabolic degeneracy, if the conical manifold

$$K(\lambda_1, \ldots, \lambda_n) = 0 \tag{1.3}$$

does not have real points apart from $\lambda_1 = 0, \ldots, \lambda_n = 0$, Eq. (1.1) is called elliptic at the point x. On this assumption we say that Eq. (1.1) is hyperbolic at the point x, if a straight line exists in the space of the variable $\lambda_1, \ldots, \lambda_n$, such that if we take it as the coordinate axis in the new variables $\mu_1, \ldots, \mu_n$, obtained using the affine transformation $\lambda_1, \ldots, \lambda_n$, then, with respect to the coordinates, which vary along this axis, the transformed equation (1.3) has exactly m real roots (simple or multiple) for any choice of values of the remaining coordinates.

We divide partial differential equations into types in a similar way in the nonlinear case - according to the properties of the corresponding characteristic form. Since the coefficients of this form depend (together with the point x) on the required solution and on its derivatives, in this case classification by type only makes sense for the solution chosen.

We will say that Eq. (1.1) is parabolic, elliptic or hyperbolic in the domain D if, at each point x of this domain it degenerates parabolically, is elliptic or hyperbolic. When Eq. (1.1) belongs to different types in the different parts of the domain D in which it is specified, it is said to be a mixed-type equation in this domain.

3. Linear second-order partial differential equations. Linear second-order partial differential equations can be written in the form

$$L(u) = \sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B^i(x) \frac{\partial u}{\partial x_i} + C(x) u = F(x). \qquad (1.4)$$

We say that Eq. (1.4) is homogeneous or nonhomogeneous in the domain D in which it is specified, depending on whether the function $F(x)$ is identically zero or is nonzero in this domain.

The expression

$$\sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

is called the principal part of the differential operator, on the left-hand side of Eq. (1.4).

When the coefficients A^{ij}, B^i, C and the right-hand side F of Eq. (1.4) are scalar functions and the required solution $u(x)$ is also scalar, the characteristic form (1.2) is quadratic:

$$Q(\lambda_1, \ldots, \lambda_n) = \sum_{i,j=1}^{n} A^{ij}(x) \lambda_i \lambda_j. \qquad (1.5)$$

At points at which the coefficients A^{ij} all equal zero, the order of Eq. (1.4) degenerates. At these points a division into types obviously makes no sense. If we eliminate the degeneracy of the order, then, using the definition given in Sect.2, Eq.(1.4) will be elliptic, hyperbolic or parabolic depending on whether the form (1.5) is definite (positive or negative), alternating or degenerate.

At each point x of the domain in which Eq.(1.4) is specified, we can use a nonsingular affine transformation of the real variables $\lambda_1, \dots, \lambda_n$

$$\lambda_i = \lambda_i(\xi_1, \dots, \xi_n), \quad i = 1, \dots, n,$$

to reduce the quadratic form (1.5) to the canonical form

$$Q = \sum_{i=1}^{n} \alpha_i \xi_i^2,$$

where the coefficients α_i, $i = 1, \dots, n$, take the values 1, −1, 0, whilst the number of negative (positive) coefficients (the inertia index) and the number of zero coefficients (the form defect) are affine invariants. In the elliptic case all $\alpha_i = 1$ or all $\alpha_i = -1$. In the hyperbolic case one of the coefficients α_i equals unity, and all the remaining ones equal minus one (or vice versa). In the parabolic case at least one of these coefficients equals zero.

The Laplace equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \qquad (1.6)$$

the wave equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 u}{\partial x_n^2} = 0 \qquad (1.7)$$

and the heat conduction equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial x_n} = 0. \qquad (1.8)$$

are typical examples of elliptic, hyperbolic and parabolic equations.

Eq. (1.4), which is elliptic in the domain D, is called uniformly elliptic, if the nonzero constants k_0 and k_1 of the same sign exist, such that

$$k_0 \sum_{i=1}^{n} \lambda_i^2 \leqslant Q(\lambda_1, \ldots, \lambda_n) \leqslant k_1 \sum_{i=1}^{n} \lambda_i^2 \tag{1.9}$$

for all points $x \in D$. The Laplace equation (1.6) is uniformly elliptic in any domain of the space E_n.

Since the characteristic quadratic form (1.5) as illustrated by the equation

$$x_n \frac{\partial^2 u}{\partial x_1^2} + \sum_{i=2}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1.10}$$

has the form

$$Q = x_n \lambda_1^2 + \sum_{i=2}^{n} \lambda_i^2, \tag{1.11}$$

this equation is elliptic in the upper half-space $x_n > 0$, hyperbolic in the lower half-space $x_n < 0$ and parabolically degenerates at all points of the hyperplane $x_n = 0$. In the domain D, which lies in the upper half-space $x_n > 0$ and is adjacent to the hyperplane $x_n = 0$, Eq. (1.10) is not uniformly elliptic because the coefficient of λ_1^2 on the right-hand side of (1.11) approaches zero as $x_n \to 0$ and therefore it is impossible to select nonzero constants k_0 and k_1 of the same sign, such that condition (1.9) holds for all the points of the domain D. Example (1.10) relates to mixed-type equations in any domain D of the space E_n, whose intersection with the hyperplane $x_n = 0$ is not empty.

In the case of the two independent variables x_1 and x_2 we shall write Eq. (1.4) and the quadratic form (1.5) in expanded form: