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Number 118

Hodge Theory, Complex Geometry, and Representation Theory

Mark Green
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Introduction

This monograph is based on ten lectures given by the second author at the CBMS sponsored conference *Hodge Theory, Complex Geometric and Representation Theory* that was held during June, 2012 at Texas Christian University, and on selected developments that have occured since then in the general areas covered by those lectures. The original material covered in the lectures and in the appendices is largely on joint work by the three authors.

This work roughly separates into two parts. One is the lectures themselves, which appear here largely as they were given at the CBMS conference and which were circulated at that time. The other part is the appendices to the later lectures. These cover material that was either related to the lecture, such as selected further background or proofs of results presented in the lectures, or new topics that are related to the lecture but have been developed since the conference. We have chosen to structure this monograph in this way because the lectures give a fairly succinct, in some places informal, account of the main subject matter. The appendices then give, in addition to some further developments, further details and proofs of several of the main results presented in the lectures.

These lectures are centered around the subjects of Hodge theory and representation theory and their relationship. A unifying theme is the geometry of homogeneous complex manifolds.

Finite dimensional representation theory enters in multiple ways, one of which is the use of Hodge representations to classify the possible realizations of a reductive, $\mathbb{Q}$-algebraic group as a Mumford-Tate group. The geometry of homogeneous complex manifolds enters through the study of Mumford-Tate domains and Hodge domains and their boundaries. It also enters through the cycle and correspondence spaces associated to Mumford-Tate domains. Running throughout is the analysis of the $G_{\mathbb{R}}$-orbit structure of flag varieties and the $G_{\mathbb{R}}$-orbit structure of the complexifications of symmetric spaces $G_{\mathbb{R}}/K$ where K contains a compact maximal torus.

Infinite dimensional representation theory and the geometry of homogeneous complex manifolds interact through the realization, due primarily to Schmid, of the Harish-Chandra modules associated to discrete series representations, especially their limits, as cohomology groups associated to homogeneous line bundles. It also enters through the work of Carayol on automorphic cohomology, which involves the Hodge theory associated to Mumford-Tate domains and to their boundary components.

Throughout these lectures we have kept the "running examples" of SL_2, $S\mathcal{U}(2,1)$, $\mathrm{Sp}(4)$ and $\mathrm{SO}(4,1)$. Many of the general results whose proofs are not given in the lectures are easily verified in the running examples. They also serve to illustrate and make concrete the general theory.

We have attempted to keep the lecture notes as accessible as possible. Both the subjects of Hodge theory and representation theory are highly developed and extensive areas of mathematics and we are only able to touch on some aspects where they are related. When more advanced concepts from another area have been used, such as local cohomology and Grothendieck duality from algebraic geometry at the end of Lecture 6, we have illustrated them through the running examples in the hope that at least the flavor of what is being done will come through.

Lectures 1 and 2 are basically elementary, assuming some standard Riemann surface theory. In this setting we will introduce essentially all of the basic concepts that appear later. Their purpose is to present up front the main ideas in the theory, both for reference and to try to give the reader a sense of what is to come. At the end of Lecture 2 we have given a more extensive summary of the topics that are covered in the later lectures and in the appendices. The reader may wish to use this as a more comprehensive introduction. Lecture 3 is essentially self-contatined, although some terminology from Lie theory and algebraic groups will be used. Lecture 4 will draw on the structure and representation theory of complex Lie algebras and their real forms. Lecture 5 will use some of the basic material about infinite dimensional representation theory and the theory of homogeneous complex manifolds. In Lectures 6 and 7 we will draw from complex function theory and, in the last part of Lecture 6, some topics from algebraic geometry. Lectures 8 and 9 will utilize the material that has gone before; they are mainly devoted to specific computations in the framework that has been established. The final Lecture 10 is devoted to issues and questions that arise from the earlier lectures.

We refer to the end of Lecture 2 for a more detailed account of the contents of the lectures and appendices.

As selected general references to the topics covered in this work we mention

- for a general theory of complex manifolds, [**Cat1**], [**Ba**], [**De**], [**GH**], [**Huy**] and [**We**];
- for Hodge theory, in addition to the above references, [**Cat2**], [**ET**], [**PS**], [**Vo1**], [**Vo2**];
- for period domains and variations of Hodge structure, in addition to the references just listed, [**CM-SP**], [**Ca**];
- for Mumford-Tate groups and domains and Hodge representation [**Mo1**], [**Mo2**], [**GGK1**] and [**Ro1**];
- for general references for Lie groups [**Kn1**] and for representation theory [**Kn2**]; specific references for topics covered in Lecture 5 are the expository papers [**Sch2**], [**Sch3**];
- for a general reference for flag varieties and flag domains [**FHW**]; [**GS1**] for an early treatment of some of the material presented below, and [**GGK2**], [**GG1**] and [**GG2**] for a more extensive discussion of some of the topics covered in this monograph;
- for a general reference for Penrose transforms [**BE**] and [**EGW**]; [**GGK2**], [**GG1**] for the material in this work;
- for mixed Hodge structures [**PS**] and [**ET**], for limiting mixed Hodge structures [**CKS1**], [**CKS2**], and [**KU**], [**KP1**] and [**KP2**] for boundary components of Mumford-Tate domains;
- for the classical theory of Shimura varieties from a Hodge-theoretic perspective [**Ke2**].

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LECTURE 1

The Classical Theory: Part I

The first two lectures will be largely elementary and expository. They will deal with the upper-half-plane $\mathcal{H}$ and Riemann sphere $\mathbb{P}^1$ from the points of view of Hodge theory, representation theory and complex geometry. The topics to be covered will be

(i) compact Riemann surfaces of genus one (= 1-dimensional complex tori) and polarized Hodge structures (PHS) of weight one;
(ii) the space $\mathcal{H}$ of PHS's of weight one and its compact dual $\mathbb{P}^1$ as homogeneous complex manifolds;
(iii) the geometry and representation theory associated to $\mathcal{H}$;
(iv) equivalence classes of PHS's of weight one, as parametrized by $\Gamma\backslash\mathcal{H}$, and automorphic forms;
(v) the geometric representation theory associated to $\mathbb{P}^1$, including the realization of higher cohomology by global, holomorphic data;
(vi) Penrose transforms in genus $g = 1$ and $g \geqq 2$.

Assumptions.

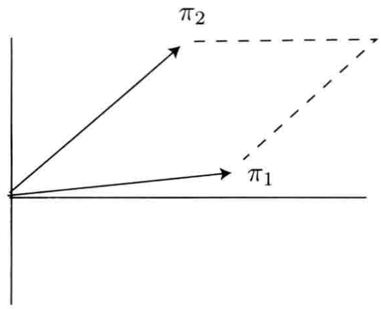
- basic knowledge of complex manifolds (in this lecture mainly Riemann surfaces);
- elementary topology and manifolds, including de Rham's theorem;
- some familiarity with classical modular forms will be helpful but not essential;[1]
- some familiarity with the basic theory of Lie groups and Lie algebras.

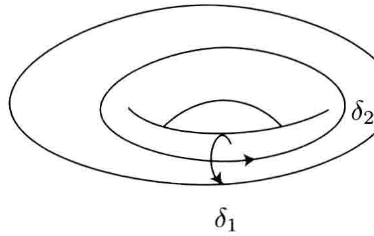
Complex tori of dimension one. We let X = compact, connected complex manifold of dimension one and genus one. Then X is a complex torus $\mathbb{C}/\Lambda$ where

$$\Lambda = \{n_1\pi_1 + n_2\pi_2\}_{n_1,n_2\in\mathbb{Z}} \subset \mathbb{C}$$

[1] The classical theory will be covered in the article **[Ke1]** by Matt Kerr in the Contemporary Mathematics volume, published by the AMS and that is associated to the CBMS conference.

is a lattice. The pictures are





Here $\delta_1 \leftrightarrow \pi_1$ and $\delta_2 \leftrightarrow \pi_2$ give a basis for $H_1(X, \mathbb{Z})$.

The complex plane $\mathbb{C} = \{z = x + iy\}$ is oriented by

$$dx \wedge dy = \left(\tfrac{i}{2}\right) dz \wedge d\bar{z} > 0.$$

We choose generators π_1, π_2 for Λ with $\pi_1 \wedge \pi_2 > 0$, and then the intersection number

$$\delta_1 \cdot \delta_2 = +1.$$

We set $V_{\mathbb{Z}} = H^1(X, \mathbb{Z})$, $V = V_{\mathbb{Z}} \otimes \mathbb{Q} = H^1(X, \mathbb{Q})$ and denote by

$$\begin{cases} Q : V \otimes V \to \mathbb{Q} \\ \mathbb{Q}(v, v') = -Q(v', v) \end{cases}$$

the cup-product, which via Poincaré duality $H_1(X, \mathbb{Q}) \cong H^1(X, \mathbb{Q})$ is the intersection form.

We have

$$\begin{array}{c} H^1(X, \mathbb{C}) \cong H^1_{\mathrm{DR}}(X) = \left\{ \begin{array}{c} \text{closed 1-forms } \psi \\ \text{modulo exact} \\ \text{1-forms } \psi = d\zeta \end{array} \right\} \\ \wr\| \\ H^1(X, \mathbb{Z})^* \otimes \mathbb{C} \end{array}$$

and it may be shown that

$$H^1_{\mathrm{DR}}(X) \cong \operatorname{span}_{\mathbb{C}} \{dz, d\bar{z}\}.$$

The pairing of cohomology and homology is given by *periods*

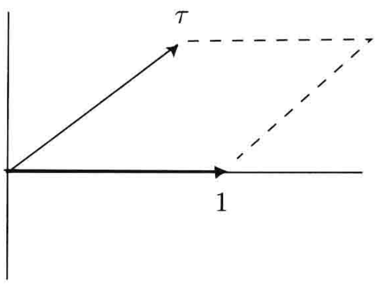
$$\pi_i = \int_{\delta_i} dz$$

and $\Pi = \binom{\pi_2}{\pi_1}$ is the *period matrix* (note the order of the π_i's).

Using the basis for $H^1(X, \mathbb{C})$ dual to the basis δ_1, δ_2 for $H_1(X, \mathbb{C})$, we have

$$\begin{array}{ccc} H^1(X, \mathbb{C}) & \cong & \mathbb{C}^2 = \text{column vectors} \\ \cup & & \cup \\ dz & = & \Pi. \end{array}$$

We may scale $\mathbb{C}$ by $z \to \lambda z$, and then $\Pi = \lambda \Pi$ so that the period matrix should be thought of as point in $\mathbb{P}^1$ with homogeneous coordinates $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$. By scaling, we may normalize to have $\pi_1 = 1$, so that setting $\tau = \pi_2$ the normalized period matrix is $\left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$ where $\operatorname{Im}\tau > 0$.



Differential forms on an n-dimensional complex manifold Y with local holomorphic coordinates $z_1, \dots, z_n$ are direct sums of those of *type* (p, q)

$$f\, \underbrace{dz_{i_1} \wedge \cdots \wedge dz_{i_p}}_{p} \wedge \underbrace{d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}}_{q}.$$

Thus the C^∞ forms of degree r on Y are

$$\begin{cases} A^r(Y) = \underset{p+q=r}{\oplus} A^{p,q}(Y) \\ A^{q,p}(Y) = \overline{A^{p,q}(Y)}. \end{cases}$$

Setting

$$H^{1,0}(X) = \operatorname{span}\{dz\}$$
$$H^{0,1}(X) = \operatorname{span}\{d\bar{z}\}$$

we have

$$\begin{cases} H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \\ H^{0,1}(X) = \overline{H^{1,0}(X)}. \end{cases}$$

This says that the above decomposition of the 1-forms on X induces a similar decomposition in cohomology. This is true in general for a compact Kähler manifold (Hodge's theorem) and is the basic starting point for Hodge theory. A recent source is [**Cat1**].

From $dz \wedge dz = 0$ and $\left(\frac{i}{2}\right) dz \wedge d\bar{z} > 0$, by using that cup-product is given in de Rham cohomology by wedge product and integration over X we have

$$\begin{cases} Q\big(H^{1,0}(X), H^{1,0}(X)\big) = 0 \\ iQ\big(H^{1,0}(X), \overline{H^{1,0}(X)}\big) > 0. \end{cases}$$

Using the above bases the matrix for Q is

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and these relations are

$$\begin{cases} Q(\Pi, \Pi) = {}^t\Pi Q \Pi = 0 \\ iQ(\Pi, \overline{\Pi}) = i\,{}^t\overline{\Pi} Q \Pi > 0. \end{cases}$$

For $\Pi = \left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$ the second is just $\operatorname{Im}\tau > 0$.

DEFINITIONS. (i) A *Hodge structure* of weight one is given by a $\mathbb{Q}$-vector space V with a line $V^{1,0} \subset V_{\mathbb{C}}$ satisfying

$$\begin{cases} V_{\mathbb{C}} &= V^{1,0} \oplus V^{0,1} \\ V^{0,1} &= \overline{V}^{1,0}. \end{cases}$$

(ii) A *polarized Hodge structure* of weight one (PHS) is given by the above together with a non-degenerate form

$$Q : V \otimes V \to \mathbb{Q}, \qquad Q(v, v') = -Q(v', v)$$

satisfying the Hodge-Riemann bilinear relations

$$\begin{cases} Q(V^{1,0}, V^{1,0}) = 0 \\ iQ(V^{1,0}, \overline{V}^{1,0}) > 0. \end{cases}$$

In practice we will usually have $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$. The reason for working with $\mathbb{Q}$ will be explained later.

When $\dim V = 2$, we may always choose a basis so that $V \cong \mathbb{Q}^2$ = column vectors and Q is given by the matrix above. Then $V^{1,0} \cong \mathbb{C}$ is spanned by a point

$$\left[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right] \in \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$$

Identification. The space of PHS's of weight one *(period domain)* is given by the upper-half-plane

$$\mathcal{H} = \{\tau : \operatorname{Im} \tau > 0\}.$$

The *compact dual* $\check{\mathcal{H}}$ given by subspaces $V^{1,0} \subset V_{\mathbb{C}}$ satisfying $Q(V^{1,0}, V^{1,0}) = 0$ (this is automatic in this case) is $\check{\mathcal{H}} = \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$ where

$$\mathbb{P}^1 = \{\tau\text{-plane}\} \cup \infty = \text{lines through the origin in } \mathbb{C}^2.[2]$$

It is well known that $\mathcal{H}$ and $\mathbb{P}^1$ are *homogeneous complex manifolds*; i.e., they are acted on transitively by Lie groups. Here are the relevant groups. Writing

$$z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \qquad w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

and using Q to identify $\Lambda^2 V$ with $\mathbb{Q}$ we have

$$Q(z, w) = {}^t w Q z = z \wedge w$$

and the relevant groups are

$$\begin{cases} \operatorname{Aut}(V_{\mathbb{R}}, Q) \cong \mathrm{SL}_2(\mathbb{R}) & \text{for } \mathcal{H} \\ \operatorname{Aut}(V_{\mathbb{C}}, Q) \cong \mathrm{SL}_2(\mathbb{C}) & \text{for } \mathbb{P}^1. \end{cases}$$

In terms of the coordinate τ the action is the familiar one:

$$\tau \to \frac{a\tau + c}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$. This is because $\tau = z_0/z_1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} az_0 + bz_1 \\ cz_0 + dz_1 \end{pmatrix} = z_1 \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}.$$

[2] **[CM-SP]** is a general reference for period domains and their differential geometric properties. A recent source is [**Ca**].

If we choose for our reference point $i \in \mathcal{H}$ $(= [\begin{smallmatrix} i \\ 1 \end{smallmatrix}] \in \mathbb{P}^1)$, then we have the identifications

$$\begin{cases} \mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \\ \mathbb{P}^1 \cong \mathrm{SL}_2(\mathbb{C})/B \end{cases}$$

where (this is a little exercise)

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : i(a-d) = -b-c \right\}.$$

The Lie algebras are (here $\mathfrak{k} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)

$$\mathrm{sl}_2(\mathfrak{k}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b \in \mathfrak{k} \right\}$$

$$\mathrm{so}(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$\mathfrak{b} = \left\{ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

REMARK. From a Hodge-theoretic perspective the above identifications of the period domain $\mathcal{H}$ and its compact dual $\check{\mathcal{H}}$ are the most convenient. From a group-theoretic perspective, it is frequently more convenient to set

$$\zeta = \frac{\tau - i}{\tau + i}, \qquad \operatorname{Im}\tau > 0 \Leftrightarrow |\zeta| < 1$$

and identify $\mathcal{H}$ with the unit disc $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$. When this is done, $\mathrm{SL}_2(\mathbb{R})$ becomes the other real form

$$SU(1,1)_{\mathbb{R}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : {}^t\bar{g}\mathbb{H}g = \mathbb{H} \right\}$$

of $\mathrm{SL}_2(\mathbb{R})$, where here $\mathbb{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\mathcal{H} \ni i \leftrightarrow 0 \in \Delta$$

$$\mathrm{SO}(2) \leftrightarrow \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$$

$$B \leftrightarrow \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \right\}.$$

Thus, for the Δ model $\mathrm{SO}(2)$ becomes a "standard" maximal torus and B is a "standard" Borel subgroup.

We now think of $\mathcal{H}$ as the parameter space for the family of PHS's of weight one and with $\dim V = 2$. Over $\mathcal{H}$ there is the natural *Hodge bundle*

$$\mathbb{V}^{1,0} \to \mathcal{H}$$

with fibres

$$\mathbb{V}^{1,0}_\tau := V^{1,0}_\tau = \text{ line in } V_{\mathbb{C}}.$$

Under the inclusion $\mathcal{H} \hookrightarrow \mathbb{P}^1$, the Hodge bundle is the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Both $\mathbb{V}^{1,0}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ are examples of homogeneous vector bundles.

In general, given

- a homogeneous space
$$Y = A/B$$
where A is a Lie group and $B \subset A$ is a closed subgroup, and
- a linear representation $r : B \to \operatorname{Aut} E$ where E is a complex vector space,

there is an associated homogeneous vector bundle

$$\begin{array}{ccc} \mathbb{E} & := & A \times_B E \\ \downarrow & & \downarrow \\ Y & = & A/B \end{array}$$

where $A \times_B E$ is the trivial vector bundle $A \times E$ factored by the equivalence relation

$$(a, e) \sim (ab, r(b^{-1})e)$$

where $a \in A$, $e \in E$, $b \in B$. The group A acts on $\mathbb{E}$ by $a \cdot (a', e) = (aa', e)$ and there is an A-equivariant action on $\mathbb{E} \to Y$. There is an evident notion of a morphism of homogeneous vector bundles; then $\mathbb{E} \to Y$ is trivial *as a homogeneous vector bundle* if, and only if, $r : B \to \operatorname{Aut}(E)$ is the restriction to B of a representation of A.

EXAMPLE. Let $\tau_0 \in \mathcal{H} \subset \mathbb{P}^1$ be the reference point. For the standard linear representation of $\mathrm{SL}_2(\mathbb{C})$ on $V_{\mathbb{C}}$, the Borel subgroup B is the stability group of the flag

$$(0) \subset V^{1,0}_{\tau_0} \subset V_{\mathbb{C}}.$$

It follows that there is over $\mathbb{P}^1$ an exact sequence of $\mathrm{SL}_2(\mathbb{C})$-homogeneous vector bundles

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{V} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

where $\mathbb{V} = \mathbb{P}^1 \times V_{\mathbb{C}}$ with $g \in \mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{V}$ by $g \cdot ([z], v) = ([gz], gv)$. The restriction to $\mathcal{H}$ of this sequence is an exact sequence of $\mathrm{SL}_2(\mathbb{R})$-homogeneous bundles

$$0 \to \mathbb{V}^{1,0} \to \mathbb{V} \to \mathbb{V}^{0,1} \to 0.$$

The bundle $\mathbb{V}^{1,0}$ is given by the representation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \to e^{i\theta}$$

of $\mathrm{SO}(2)$. Using the form Q the quotient bundle $\mathbb{V}/\mathbb{V}^{1,0} := \mathbb{V}^{0,1}$ is identified with the dual bundle $\mathbb{V}^{1,0*}$.

The *canonical line bundle* is

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Thus

$$\omega_{\mathcal{H}} \cong (\mathbb{V}^{1,0})^{\otimes 2}.$$

Proof. For the Grassmannian $Y = \mathrm{Gr}(n, E)$ of n-planes P in a vector space E there is the standard $\mathrm{GL}(E)$-equivariant isomorphism

$$T_P Y \cong \operatorname{Hom}(P, E/P).$$

In the case above where $E = \mathbb{C}^2$ and $z = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right] \in \mathbb{P}^1$ we have

$$T_z \mathbb{P}^1 \cong V_z^{1,0^*} \otimes V_{\mathbb{C}}/V_z^{1,0}$$

where $V_z^{1,0}$ is the line in $V_{\mathbb{C}}$ corresponding to z. If we use the group $\mathrm{SL}_2(\mathbb{C})$ that preserves Q in place of $\mathrm{GL}_2(\mathbb{C})$, then

$$V_{\mathbb{C}}/V_z^{1,0} \cong V_z^{1,0^*}.$$

Thus the cotangent space

$$T_z^*\mathbb{P}^1 \cong V_z^{2,0}$$

where in general we set $\mathbb{V}^{n,0} = (\mathbb{V}^{1,0})^{\otimes n}$. The above identification $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ is an $\mathrm{SL}_2(\mathbb{C})$, but *not* $\mathrm{GL}_2(\mathbb{C})$, equivalence of homogenous bundles.

Convention. We set

$$\omega_{\mathcal{H}}^{1/2} = \mathbb{V}^{1,0}.$$

The Hodge bundle $\mathbb{V}^{1,0} \to \mathcal{H}$ has an $\mathrm{SL}_2(\mathbb{R})$-invariant metric, *the Hodge metric*, given fibrewise by the 2nd Hodge-Riemann bilinear relation. The basic invariant of a metric is its *curvature*, and we have the following

General fact. Let $\mathbb{L} \to Y$ be an Hermitian line bundle over a complex manifold Y. Then the *Chern* (or curvature) *form* is

$$c_1(\mathbb{L}) = \left(\frac{1}{2\pi i}\right) \partial\overline{\partial} \log \|s\|^2$$

where $s \in \mathcal{O}(\mathbb{L})$ is any non-vanishing local holomorphic section and $\|s\|^2$ is its length squared.

Basic calculation.

$$c_1(\mathbb{V}^{1,0}) = \frac{1}{4\pi}\frac{dx \wedge dy}{y^2} = \frac{i}{2\pi}\frac{d\tau \wedge \overline{d\tau}}{(\operatorname{Im}\tau)^2}.$$

This has the following

Consequence. *The tangent bundle*

$$T\mathcal{H} \cong \mathbb{V}^{0,2}$$

has a metric

$$ds^2_{\mathcal{H}} = \frac{dx^2 + dy^2}{y^2} = \left(\frac{1}{(\operatorname{Im}\tau)^2}\right) \operatorname{Re}(dz\, d\bar{z})$$

of constant negative Gauss curvature.

Before giving the proof we shall make a couple of observations.

Any $\mathrm{SL}_2(\mathbb{R})$ invariant Hermitian metric on $\mathcal{H}$ is conformally equivalent to $dx^2 + dy^2$; hence it is of the form

$$h(x,y)\left(\frac{dx^2 + dy^2}{y^2}\right)$$

for a positive function $h(x,y)$. Invariance under translation $\tau \to \tau + b$, $b \in \mathbb{R}$, corresponding to the subgroup $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, implies that $h(x,y) = h(y)$ depends only on y. Then invariance under $\tau \to a\tau$ corresponding to the subgroup $\left(\begin{smallmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{smallmatrix}\right)$, $a > 0$, gives that $h(y) =$ constant. A similar argument gives that $c_1(\mathbb{V}^{1,0})$ is a constant multiple of the form above.

The all important sign of the curvature K may be determined geometrically as follows: Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete group such that $Y = \Gamma\backslash\mathcal{H}$ is a compact

Riemann surface of genus $g \geqq 2$ with the metric induced from that on $\mathcal{H}$. By the Gauss-Bonnet theorem

$$0 > 2 - 2g = \chi(Y) = \frac{1}{4\pi}\int_Y K dA = K\left(\frac{\text{Area}(Y)}{4\pi}\right).$$

PROOF OF BASIC CALCULATION. We define a section $s \in \Gamma(\mathcal{H}, \mathbb{V}^{1,0})$ by

$$s(\tau) = \begin{pmatrix}\tau\\1\end{pmatrix} \in \mathbb{V}^{1,0}_\tau.$$

The length squared is given by

$$\|s(\tau)\|^2 = i\,{}^t\overline{s(\tau)}Qs(\tau) = 2y.$$

Using for $\tau = x + iy$

$$\begin{cases}\partial_\tau = \frac{1}{2}(\partial_x - i\partial_y)\\ \partial_{\bar\tau} = \frac{1}{2}(\partial_x + i\partial_y)\end{cases}$$

we obtain

$$\frac{i}{2\pi}\overline{\partial}\partial = -\frac{1}{4\pi}(\partial_x^2 + \partial_y^2)dx \wedge dy.$$

This gives

$$\frac{i}{2\pi}\overline{\partial}\partial \log \|s(\tau)\|^2 = \frac{1}{4\pi}\frac{dx \wedge dy}{y^2}.$$

REMARK. There is also a $S\mathcal{U}(2)$-invariant metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$ induced from the standard metric on $\mathbb{C}^2$. For this metric

$$\|s(\tau)\|_c^2 = 1 + |\tau|^2$$

(the subscript c on $\|\ \|_c^2$ stands for "compact"). Then we have

$$c_1(\mathbb{V}_c^{1,0}) = -\frac{1}{4\pi}\frac{dx \wedge dy}{(1+|\tau|^2)^2}.$$

Thus, $\mathbb{V}^{1,0} \to \mathcal{H}$ is a *positive* line bundle whereas $\mathbb{V}_c^{1,0} \to \mathbb{P}^1$ is a *negative* line bundle with

$$\deg \mathcal{O}_{\mathbb{P}^1}(-1) = \int_{\mathbb{P}^1} c_1(\mathbb{V}_c^{1,0}) = -1.$$

This *sign reversal* between the $\mathrm{SL}_2(\mathbb{R})$-invariant curvature on the open domain $\mathcal{H}$ and the $S\mathcal{U}(2)$ (= compact form of $\mathrm{SL}_2(\mathbb{C})$)-invariant metric on the compact dual $\check{\mathcal{H}} = \mathbb{P}^1$ will hold in general and is a fundamental phenomenon in Hodge theory.

Above we have holomorphically trivialized $\mathbb{V}^{1,0} \to \mathcal{H}$ using the section

$$s(\tau) = \begin{pmatrix}\tau\\1\end{pmatrix}.$$

We have also noted that we have the isomorphism of $\mathrm{SL}_2(\mathbb{R})$-homogeneous line bundles

$$\omega_{\mathcal{H}} \cong \mathbb{V}^{2,0}.$$

Now $\omega_{\mathcal{H}}$ has a section $d\tau$ and a useful fact is that under this isomorphism

$$d\tau = s(\tau)^2.$$