INTRODUCTION TO THE CALCULUS

BY

WILLIAM F. OSGOOD, PH.D., LL.D

PERKINS PROFESSOR OF MATHEMATICS
IN HARVARD UNIVERSITY

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PREFACE

THE present book is a revision of the author's *A First Course in the Differential and Integral Calculus.* The plan of treatment is essentially the same, but the presentation is fuller, and the lists of exercises have been enlarged by problems of value to the student of good average ability.

The object of the book is two-fold; namely, to set forth the application of the calculus to problems of geometry and physics of the first order of importance, and to make clear the thought which underlies the calculus.

To attain the first end, the physical picture must be shown to the student who has no technical knowledge of physics, but who can understand the simplest concepts of that science when clearly presented to him. Consequently, great care has been taken each time that a new physical notion has been introduced to say exactly what is meant, and then to show precisely how mathematics applies to the situation in hand.

On the other hand, thorough training in the formal part of the calculus is essential if the student is to develop power in the use of his tools, and exercises adequate for this purpose have been included in lists properly graded in point of difficulty.

Behind and beneath it all is the idea of the limit. Abstract discussions of this idea are not in place in an elementary treatment. The beginner comes to assimilate the method of limits by seeing it applied, with such details as have a meaning for him, in proving the few fundamental theorems on which the calculus rests, and in formulating geometrical and physical problems.

The treatment is flexible from start to finish. The teacher can go as far, or stop as early, as he pleases in presenting the material of a given chapter. Thus in the chapter on Definite Integrals any reasonable selection from the topics there treated can be made, the order changed, and whole paragraphs omitted without marring the unity of the course. The same is true of the chapters on Mechanics and Infinite Series. Many of these abridged treatments are altogether admirable; but no one of them can be expected to appear to any large body of teachers as preeminently the best. It is primarily a question of the personal equation of the teacher himself. The book also takes account of the personal equation of the student. A skilful teacher will help his best students to see as far and as deeply as their talents permit. He can do this with this text without losing by the way the less gifted students; for each time that the scene changes the new subject is presented with the utmost simplicity.

The book is intended alike for the engineer or the physicist and for the student of pure mathematics. The best methods of the present day in the calculus, when properly presented, are within the reach of the former student and afford him most valuable tools for the understanding of his own technical problems. On the other hand, the student of pure mathematics cannot do better than early to inform himself concerning those relations of the calculus to physics, to which this great branch of mathematics owes its origin.

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CHAPTER I

INTRODUCTION

THE Calculus was invented in the seventeenth century by the mathematician, astronomer, and physicist, Sir Isaac Newton in England, and the philosopher Leibniz in Germany. The reaction of the invention on geometry and mathematical physics was most important. In fact, by far the greatest part of the mathematics and the physics of the present day owes its existence to this invention.

1. Functions. The word *function*, in mathematics, was first applied to an expression involving one or more letters which represent variable quantities; as, for example, the expressions

(*a*) $$x^3, \qquad 2x^3 - 3x + 1;$$

(*b*) $$\sqrt{x}, \qquad \sqrt{a^2 - x^2};$$

(*c*) $$\frac{x^2}{a+x}, \qquad \frac{xy}{x^2+y^2}, \qquad \frac{ax+by}{\sqrt{x^2+y^2+z^2}};$$

(*d*) $$\sin x, \qquad \log x, \qquad \tan^{-1} x.$$

In the second example under (*b*), two letters enter; but a is thought of as chosen in advance and then held fast, x alone being variable. A quantity of this kind is called a *constant*. Thus

$$ax + b$$

is a function of x which depends on two constants, a and b.

Such expressions are written in symbolic, or abbreviated, form as $f(x)$, $f(x, y)$ (read: "f of x," "f of x and y" etc.); other letters in common use being F, ϕ, Φ, etc.* Thus the equation

(1) $$f(x) = 2x^3 - 3x + 1$$

defines the function $f(x)$ in the present case to be $2x^3-3x+1$. Again,

(2) $$\phi(x, y, z) = x^2 + y^2 + z^2$$

is an equation defining the function $\phi(x, y, z)$ as $x^2 + y^2 + z^2$.

We shall be concerned for the present with functions of one single variable, as illustrated by (1) above. Here, x is called the *independent* variable, since we assign to it any value we like. The value of the function, or more briefly, the *function*, is called the *dependent variable*, and is often denoted by a single letter, as

$$y = f(x)$$

or

$$y = 2x^3 - 3x + 1.$$

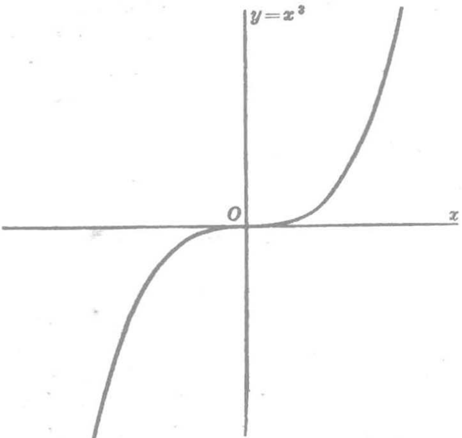

FIG. 1

Graphs. A function of a single variable,

$$y = f(x),$$

can be represented geometrically by its graph, and this representation is of great aid in studying the properties of the function. The independent variable is laid off as the x-coordinate, or abscissa, and the dependent variable, or func-

* To distinguish between $f(x)$ and $F(x)$, read the first "small f of x" and the second, "large F of x."

tion, as the y-coordinate, or ordinate. Thus the graph of the function

$$f(x) = x^3$$

is the curve

$$y = x^3.$$

Illustrations from Geometry and Physics. The familiar formulas of geometry and physics afford simple examples of functions. Thus the area, A, of a circle is given by the formula

$$A = \pi r^2,$$

where r denotes the radius, π being the fixed number 3.1416. Here, r is thought of as the independent variable, — it may have any positive value whatever, — and A is the function, or dependent variable.

Again, for the three round bodies, the volumes are:

(a) $V = \frac{4}{3}\pi r^3,$ sphere;

(b) $V = \pi r^2 h,$ cylinder;

(c) $V = \frac{\pi}{3} r^2 h,$ cone.

In (b) and (c), h denotes the altitude and r, the radius of the base; V is here a function of the *two* independent variables, r and h.

The surfaces of these bodies are given by the formulas:

(α) $S = 4\pi r^2,$ sphere;

(β) $S = 2\pi rh,$ cylinder;

(γ) $S = \pi rl,$ cone;

l, in the last formula, denoting the slant height. Thus we have three further examples of functions of one or of two variables.

The formula for a freely falling body is

$$s = \tfrac{1}{2} gt^2,$$

where s denotes the distance fallen and t the time; g is a constant, for it has just one value after the units of time and length have been chosen. Here, t is the independent variable and s is the function. If, however, we solve this equation for t:

$$t = \sqrt{\frac{2s}{g}},$$

then s becomes the independent variable and t, the function.

Sometimes two variables are connected by an equation, as

$$pv = c,$$

where p denotes the pressure of a gas and v its volume, the temperature remaining constant. Here, either variable can be chosen as the independent variable, and when the equation is solved for the other variable, the latter becomes the dependent variable, or function. Thus, if we write

$$v = \frac{c}{p},$$

p is the independent variable, and v is expressed as a function of p. But if we write

$$p = \frac{c}{v},$$

the rôles are reversed.

The Independent Variable Restricted. Often the independent variable is restricted to a certain interval, as in the case of the function

$$y = \sqrt{a^2 - x^2}.$$

Here, x must lie between $-a$ and a:

$$-a \leqq x \leqq a,$$

since other values of x make $a^2 - x^2$ negative, and the above expression has no meaning.

This was also the case with the geometric examples above cited. There, r, h, l were necessarily positive, since there is no such thing, for example, as a sphere of zero or negative radius.

The independent variable may also be restricted to being a positive whole number, as in the case of the sum of the first n

terms of a geometric progression:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Here,

$$s_n = \frac{a - ar^n}{1 - r}.$$

Suppose $a = 1$, $r = \frac{1}{2}$, the progression thus becoming

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Then

$$s_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}},$$

and we have an example of a function with the independent variable a natural number, *i.e.* a positive integer.

In the case of the functions treated in the calculus, the domain of the independent variable is a *continuum, i.e.*, for functions of a single variable, an interval, as

$$a \leqq x \leqq b, \qquad \text{or} \qquad 0 < x.$$

Ordinarily, the later letters of the alphabet, particularly x, y, z, are used to represent variables, the early letters denoting constants. Thus it will be understood, when such an expression as

$$ax^2 + bx + c$$

is written down, that a, b, c are constants and x is the variable.

Multiple-Valued Functions; Principal Value. The expressions above cited are all examples of *single-valued* functions; *i.e.* to each value of the independent variable x corresponds but one value of the function. A function may, however, be *multiple-valued;* as in the case of the function y defined by the equation

$$x^2 + y^2 = a^2.$$

Here

$$y = \pm\sqrt{a^2 - x^2},$$

and so is a double-valued function. This function is, however, completely represented by means of the two single-valued functions,

$$y = \sqrt{a^2 - x^2} \qquad \text{and} \qquad y = -\sqrt{a^2 - x^2}.$$

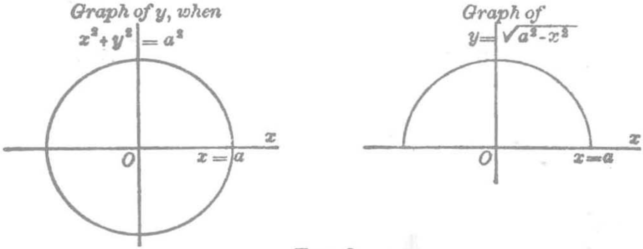

FIG. 2

They form the *branches* of this multiple-valued function.

The student should notice that the radical sign $\sqrt{\ }$ is defined as meaning the *positive* square root, NOT *either* the positive *or* the negative square root at pleasure. If it is desired to express the negative square root, the minus sign must be written in front of the radical sign, $-\sqrt{\ }$. Thus $\sqrt{4} = 2$, and not -2. This does not mean that 4 has only one square root. It means that the notation $\sqrt{4}$ calls for the positive, and not for the negative, of these two roots.

Again,

$$\sqrt{(-2)^2} = 2,$$

and not -2. For $(-2)^2 = 4$, and $\sqrt{\ }$ means the positive root. And, generally,

(1) $$\begin{cases} \sqrt{x^2} = x, & \text{if } x \text{ is positive;} \\ \sqrt{x^2} = -x, & \text{if } x \text{ is negative.} \end{cases}$$

A similar remark applies to the symbol $\sqrt[2n]{\ }$, which is likewise used to mean the positive $2n$th root. Moreover,

$$a^{\frac{1}{2}} = \sqrt{a}, \qquad a^{\frac{1}{2n}} = \sqrt[2n]{a}.$$

The function

$$y = \sqrt{x}$$

is often called the *principal value* of the double-valued function defined by the equation

$$y^2 = x.$$

Since multiple-valued functions are studied by means of single-valued functions, it will be understood henceforth, unless the contrary is explicitly stated, that the word *function* means *single-valued* function.

Absolute Value. It is frequently desirable to use merely the *numerical*, or *absolute value* of a quantity, and to have a notation for the same. The notation is: $|x|$, read "absolute value of x." Thus

$$|-3| = 3 \qquad \text{and} \qquad |3| = 3.$$

We can now write in a single formula what was formerly stated by the two equations (1), namely the definition of the radical sign, $\sqrt{\ }$:

$$\sqrt{a^2} = |a|. \tag{2}$$

Again, by the *difference* of two numbers we often mean the value of the larger less the smaller. Thus the difference of 4 and 10 is 6; and the difference of 10 and 4 is also 6. The difference of a and b, in this sense, can be expressed as either

$$|b-a| \qquad \text{or} \qquad |a-b|.$$

Continuous Functions. A function, $f(x)$, is said to be *continuous* if a slight change in x produces but a slight change in the value of the function. Thus the polynomials are readily shown to be continuous; cf. Chap. II, § 5, and all the functions with which we shall have to deal are continuous, save at exceptional points.

As an example of a function which is discontinuous at a certain point may be cited the function (see Fig. 3)

$$f(x) = \frac{1}{x}.$$