# Graduate Texts in Mathematics

*Reading in Mathematics*

Wolfgang Walter

# Ordinary Differential Equations

常微分方程

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# Preface

The author's book on *Gewöhnliche Differentialgleichungen* (Ordinary Differential Equations) was published in 1972. The present book is based on a translation of the latest, 6th, edition, which appeared in 1996, but it also treats some important subjects that are not found there. The German book is widely used as a textbook for a first course in ordinary differential equations. This is a rigorous course, and it contains some material that is more difficult than that usually found in a first course textbook; such as, for example, Peano's existence theorem. It is addressed to students of mathematics, physics, and computer science and is usually taken in the third semester. Let me remark here that in the German system the student learns calculus of one variable at the gymnasium[1] and begins at the university with a two-semester course on real analysis which is usually followed by ordinary differential equations.

**Prerequisites.** In order to understand the main text, it suffices that the reader have a sound knowledge of calculus and be familiar with basic notions from linear algebra. For complex differential equations, some facts about holomorphic functions and their integrals are required. These are summarized at the beginning of § 8 and more fully described and partly proved in part C of the Appendix. Functional analysis is developed in the text when needed. In several places there are sections denoted as Supplements, where more special subjects are treated or the theory is extended. More advanced tools such as Lebesgue's theory of integration or Schauder's fixed point theorem are occasionally used in those sections. The supplements and also § 13 can be omitted in a first reading.

**Outline of contents.** The book treats significantly more topics than can be covered in a one-semester course. It also contains material that is seldom found in textbooks and—what is perhaps more important—it uses new proofs for basic theorems. This aspect of the book calls for a closer look at contents and methods with emphasis on those places where we depart from the mainstream.

The first chapter treats classical cases of first order equations that can be solved explicitly. By means of a number of examples the student encounters the essential features of the initial value problem such as uniqueness and nonuniqueness, maximal solutions in the case of nonuniqueness, and continuous dependence on initial values in the small, but not in the large; see 1.VI–VIII. The

[1]In the German school system, the gymnasium is an academic high school that prepares students for study at the university.

phase plane and phase portraits are explained in 3.VI–VIII.

The theory proper starts with Chapter II. In this and the following chapter the initial value problem is treated first for one equation and then for systems of equations. The repetition caused by this separation of cases is minimal since all proofs carry over, while the student has the benefit that the reasoning is not burdened by technicalities about vector functions. The complex case, where the solutions are holomorphic functions, is treated in § 8; the proofs follow the pattern set in § 6 for the real case. The theory of differential inequalities in § 9 is one-dimensional by its very nature. An extension to $n$ dimensions leads to new phenomena that are treated in Supplement I of § 10.

Chapter IV is devoted to linear systems and linear differential equations of higher order. In a Supplement to § 18 the Floquet theory for systems with periodic coefficients is presented.

Linear systems in the complex domain is the topic of Chapter V. The main properties of systems with isolated singularities are developed in a novel way (see below). Equations of mathematical physics are discussed in § 25.

The main subject of Chapter VI is the Sturm–Liouville theory of boundary value and eigenvalue problems. Nonlinear boundary value problems and corresponding existence, uniqueness, and comparison theorems are also treated. In § 28 the eigenvalue theory for compact self-adjoint operators in Hilbert space is developed and applied to the Sturm–Liouville eigenvalue problem.

The last chapter deals with stability and asymptotic behavior of solutions. The linearization theorem of Grobman–Hartman is given without proof (the author is still looking for a really good proof). The method of Lyapunov is developed and applied in § 30.

An appendix consisting of four parts A (topology), B (real analysis), C (complex analysis), and D (functional analysis) contains notions and theorems that are used in the text or can lead to a deeper understanding of the subject. The fixed point theorems of Brouwer and Schauder are proved in B.V and D.XII.

In closing this overview, we point out that applications, mostly from mechanics and mathematical biology, are found in many places. Exercises, which range from routine to demanding, are dispersed throughout the text, some with an outline of the solution. Solutions of selected exercises are found at the end of the book.

**Special Features.** Two general themes exercise a profound influence throughout the book: functional analysis and differential inequalities.

**Functional Analysis.** The *contraction principle*, that is, the fixed point theorem for contractive mappings in a Banach space, is at the center. This theorem has all necessary properties to make it a fundamental principle of analysis: It is elementary, widely applicable, and far-reaching.[2] Its flexibility in connection with our subject comes to light when appropriate weighted maximum norms

[2] A remarkable theorem of Bessaga (1959) sheds light on the versatility of the contraction principle. Consider a map $T : S \to S$, where $S$ is an arbitrary set, and assume that $T$ has a unique fixed point which is also the only fixed point of $T^2, T^3, \ldots$. Then there is a metric on $S$ that makes $S$ a complete metric space and $T$ a contraction. One can even find metrics for which the Lipschitz constant of $T$ is arbitrarily small.

are used. A first example is found in the dissertation of Morgenstern (1952); references to later authors in the literature are historically unjustified. In linear complex systems, the weighted maximum norm in 21.II leads to global existence without using analytic continuation and the monodromy theorem. Moreover, this proof gives the growth properties of solutions that are needed in the treatment of singular points. The theorems on continuous dependence on initial values and parameters and on holomorphy with regard to complex parameters follow directly from the contraction principle, a fact which is still little known. Differentiability with respect to real parameters requires Ostrowski's theorem on approximate iteration 13.IV.

In the treatment of linear systems with weakly singular points, the crucial convergence proofs are also reduced to the contraction principle in a suitable Banach space.[3] For holomorphic solutions, i.e., power series expansions, this method was discovered by Harris, Sibuya, and Weinberg (1969). The logarithmic case can also be treated along these lines. This approach leads also to theorems of Lettenmeyer and others, which are beyond the scope of this book; cf. the original work cited above.

A theorem in Appendix D.VII, which is partly due to Holmes (1968), establishes a relation between the norm of a linear operator and its spectral radius. As explained in Section D.IX, this result gives a better insight into the role of weighted maximum norms.

**Differential Inequalities.** The author, who also wrote the first monograph on differential inequalities (1964, 1970), has encountered many instances where authors are unaware of basic theorems on differential inequalities that would have made their reasoning much simpler and stronger. The distinction between weak and strong inequalities is a matter of fundamental importance. In partial differential equations this is common knowledge: weak maximum or comparison principles versus strong principles of this type. Not so in ordinary differential equations. Theorem 9.IX is a strong comparison principle that prescribes precisely the occurrence of strict inequalities, while most (all?) textbooks are content with the weak "less than or equal" statement. This principle is essential for our treatment of the Sturm–Liouville theory via Prüfer transformation. Its usefulness in nonlinear Sturm theory can be seen from a recent paper, Walter (1997).

Supplement I in § 10 brings the two basic theorems on systems of differential inequalities, (i) the comparison theorem for quasimonotone systems, and (ii) Max Müller's theorem for the general case. Both were found in the mid twenties. ***Quasimonotonicity*** is a necessary and sufficient condition for extending the classical theory (including maximal and minimal solutions) from one equation to systems of equations. More recently, both theorems (i) and (ii) have been applied to population dynamics, but it is not generally known that results on

[3]The Banach space $H_\delta$ of 24.I, which is indeed a Banach algebra, can be used for a short and elegant proof of two fundamental theorems for functions of several complex variables, the preparation theorem and the division theorem of Weierstrass. This proof has been propagated by Grauert and Remmert since the sixties and can be found, e.g., in their book *Coherent Analytic Sheaves* (Grundlehren 265, Springer 1984); cf. Walter (1992) for other applications.

invariant rectangles are special cases of Müller's theorem. Theorem 10.XII is the strong version of (i); it contains M. Hirsch's theorem on strongly monotone flows, cf. Hirsch (1985) and Walter (1997).

A Supplement to § 26 describes a new approach to minimum principles for boundary value problems of Sturmian type that applies also to nonlinear differential operators; cf. Walter (1995). The strong minimum principle is generalized in 26.XIX, so that it includes now the first eigenvalue case.

In Supplement II of § 26 on nonlinear boundary value problems the method of upper and lower solutions for existence and Serrin's sweeping principle for uniqueness are presented.

**Miscellaneous Topics.** *Differential equations in the sense of Carathéodory.* The initial value problem is treated in Supplement II of § 10 and a Sturm–Liouville theory under Carathéodory assumptions in 26.XXIV and 27.XXI. As a rule, the earlier proofs for the classical case carry over. This applies in particular to the strong comparison theorem 10.XV and the strong minimum principle in 26.XXV.

*Radial solutions of elliptic equations.* This subject plays an active role in recent research on nonlinear elliptic problems. The radial $\Delta$-operator is an operator of Sturm–Liouville type with a singularity at 0. The corresponding initial value problem is treated in a supplement of § 6, and the eigenvalue problem and nonlinear boundary value problems for the unit ball in $\mathbb{R}^n$ (for radial solutions) in a Supplement to § 27.

*Separatrices* is the theme of a Supplement in § 9. Differential inequalites are essential for proving existence and uniqueness.

*Special Applications.* We mention the generalized logistic equation in a supplement to § 2, general predator–prey models in 3.VII, delay-differential equations in 7.XIV-XV, invariant sets in 10.XVI and the rubber band as a model for nonlinear oscillations in a nonsymmetric mechanical system in 11.X.

*Exact Numerics.* We give examples in which a combination of a numerical procedure and a sup-superfunction technique allows a mathematically exact computation of special values. The numerical part is based on an algorithm, developed by Rudolf Lohner (1987, 1988), that computes exact enclosures for the solutions of an initial value problem. In blow-up problems one obtains rather sharp enclosures for the location of the asymptote of the solutions; cf. 9.V. A different kind of sub- and supersolutions is used to compute a separatrix; in general, a separatrix is an unstable solution.

**Acknowledgments.** It is a pleasure to thank all those who have contributed to the making of this volume. The translator, Professor Russell Thompson, worked with expertise and patience in the face of changes and additions during the translation and furnished beautiful figures. He also suggested an improved division into chapters. Irene Redheffer acted as a mediator between author and translator with exceptional care and insight and translated the Solutions section. Her help and advice and that of Professor Ray Redheffer were indispensable. My sincere thanks go to all of them and also to other helping hands and minds.

*Karlsruhe, August 1997* Wolfgang Walter

# Note to the Reader

In references to another paragraph, the number of the paragraph is given before the number of the formula, theorem, lemma,.... For example, formula (7) in § 15 is denoted as (15.7), and theorem 15.III or corollary 15.III refers to the theorem or corollary in section III of § 15. But when citing within § 15, we write simply formula (7), Theorem III, and Corollary III. A reference to B.V refers to Section V in Part B of the Appendix.

When the name of an author is followed by the year of publication, as in Perron (1926), the source is found in the bibliography at the end of the book. My two books on analysis are cited as Walter 1 and Walter 2. A compilation of general notions and a list of symbols are found at the end of the book.

The German word *Ansatz* is used repeatedly; a footnote in Part II of the introduction gives an explanation.

# Table of Contents

# Introduction

A differential equation is an equation containing independent variables, functions, and derivatives of functions. The equation

$$y' + 2xy = 0 \tag{1}$$

is a differential equation. Here $x$ is the independent variable and $y$ is the unknown function. A *solution* is a function $y = \phi(x)$ that satisfies (1) identically in $x$, that is, $\phi'(x) + 2x \cdot \phi(x) \equiv 0$. It is easy to check that the function $y = \mathrm{e}^{-x^2}$ is a solution of (1):

$$\frac{d}{dx}(\mathrm{e}^{-x^2}) + 2x\mathrm{e}^{-x^2} \equiv 0 \quad \text{for} \quad -\infty < x < \infty.$$

We will see later that the collection of all solutions of (1) can be written in the form $y = C \cdot \mathrm{e}^{-x^2}$, where $C$ runs through the set of real numbers.

Equation (1) is a *differential equation of first order.* The general differential equation of first order has the form

$$F(x, y, y') = 0. \tag{2}$$

A function $y = y(x)$ is called a *solution* of (2) in an interval $J$ if $y(x)$ is differentiable in $J$ and

$$F(x, y(x), y'(x)) \equiv 0 \quad \text{holds for all} \quad x \in J.$$

If a differential equation contains higher order derivatives, say up to $n$th order, then the equation is called an *nth order differential equation.* Such an equation can always be written in the general form

$$F(x, y, y', \dots, y^{(n)}) = 0. \tag{3}$$

Here a *solution* is defined to be an $n$-times differentiable function such that equation (3) is satisfied identically when $y(x)$ and its derivatives are substituted into $F$. A differential equation of $n$th order is called *explicit* if it has the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}); \tag{4}$$

otherwise it is called *implicit.* For a first order ordinary differential equation the explicit form is

$$y' = f(x, y). \tag{5}$$

The above comments apply to *ordinary differential equations*, that is, to differential equations for functions $y(x)$ of a *single* independent variable $x$. If several independent variables and hence also partial derivatives are present, then the equation is called a *partial differential equation.* For example,

$$u_x + u_y = x + y$$

is a partial differential equation of first order for an unknown function $u(x, y)$. The function $u(x, y) = xy$ is a particular solution to this equation. An important example of a second order partial differential equation is the *potential equation* in three-space

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = 0,$$

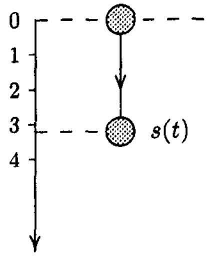
where $u = u(x, y, z)$.

In this book we will be concerned only with ordinary differential equations. The primary emphasis will be on differential equations in the real domain where the independent variable $x$ is a real variable and $y(x)$ is a real function. However, the fundamental facts about differential equations in the complex domain will also be treated.

The expression *integral of a differential equation* is another term used for a *solution*, and the terms *solution curve* and *integral curve* are used to emphasize the geometric interpretation of a solution as a curve. A family of functions $y(x; C_1, \ldots, C_n)$, depending on $x$ and $n$ parameters $C_1, \ldots, C_n$ (which vary in a point set $M \subset \mathbb{R}^n$), is called a *complete integral* or a *general solution* of the $n$th order differential equation (4) if it satisfies the following two requirements: first, each function $y(x; C_1, \ldots, C_n)$ is a solution to the differential equation (4) for an arbitrary choice of the parameters $(C_1, \ldots, C_n) \in M$, and second, all solutions can be obtained in this manner. The notion of a general solution does not play a major role in the theory of differential equations. It is used here in connection with simple examples, where it is actually possible to give all solutions explicitly in a form depending on $n$ parameters.

Differential equations play a cardinal role in the natural sciences and technology, especially in physics, for the simple reason that many physical laws take the form of a differential equation. Differential equations also appear in other scientific domains where mathematical models and theories are used. The three examples that follow are intended to give a first impression of the type of problems that arise. They all deal with the motion of a body in a gravitational field.

**I. Free Fall.** When a body at rest is suddenly released, it falls downward under the influence of gravity. This motion can be described mathematically by a function $s = s(t)$ which gives the distance that the body (or more exactly, its center of mass) has traveled up to time $t$. Other quantities of interest that can be derived from $s$ include the instantaneous velocity $v(t) = \dfrac{d}{dt}s(t) = \dot{s}(t)$ and the acceleration $a(t) = \dfrac{d}{dt}v(t) = \ddot{s}(t)$. (When describing processes in which




the independent variable represents time, it is customary to denote the independent variable by $t$ instead of $x$, and derivatives by dots instead of primes.) We learn in elementary mechanics that the acceleration of such bodies may be assumed to be constant, in fact, equal to the acceleration $g$ due to gravity at the earth's surface. Thus the distance-time function $s(t)$ satisfies the second order differential equation

$$\ddot{s} = g. \tag{6}$$

It is easy to find all of the solutions here. Indeed, it follows from integrating the equation $\dot{v}(t) = g$ that $v(t) = gt + C_1$, and likewise from $\dot{s}(t) = gt + C_1$ that

$$s(t) = \frac{1}{2}gt^2 + C_1t + C_2 \qquad (C_1,\ C_2 \text{ constant}).$$

We have thus found the complete integral of the differential equation (6).

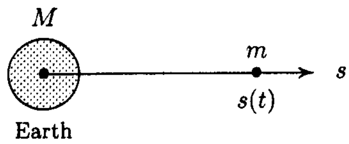
To go from this family of $\ddot{s} = g$ to the solution that corresponds to a particular physical process requires some additional information, the so-called *initial conditions*. Let us assume, for instance, that in the example above the body is at rest and is then released at time $t = 0$. Corresponding initial conditions are given by $s(0) = 0$ and $\dot{s}(0) = v(0) = 0$. From the first of these conditions it follows that $C_2 = 0$, from the second that $C_1 = 0$, and in this manner one obtains the solution

$$s(t) = \frac{1}{2}gt^2.$$

Other initial conditions lead in a like manner to other solutions.

**II. Free Fall from a Large Distance.** Now suppose that the body is at a large distance from the earth. The assumption of constant gravitational acceleration made in I is valid only near the surface of the earth. According to Newton's law of gravitation, two bodies a distance $s$ apart with masses $M$ (earth) and $m$ (test body) attract each other with a force equal to $K = \gamma\frac{Mm}{s^2}$, where $\gamma$ is the gravitational constant. By Newton's second law the acceleration now satisfies the equation

$$\ddot{s} = -\gamma M \cdot \frac{1}{s^2}. \tag{7}$$




The minus sign on the right-hand side indicates that the direction of the force is opposite to the positive $s$-direction. This differential equation of second order is significantly more difficult to integrate than equation (6). Nonetheless, the solutions can be given explicitly; we will return to this later in §11.XII. Suppose that at time $t = 0$ a test body is located a distance $R$ from the earth's center and released at rest. Then one has for initial conditions $s(0) = R$, $\dot{s}(0) = 0$.

A simple and sometimes successful method of finding solutions to a differential equation is to look for a likely "ansatz"[4] (possibly containing parameters) and to investigate whether it leads to a solution. We will try this approach in the case of equation (7) using the ansatz

$$s(t) = a \cdot t^b.$$

When this function is substituted into equation (7), the result is

$$ab(b-1)t^{b-2} = -\gamma M a^{-2} t^{-2b}.$$

Equating exponents and coefficients leads to $b - 2 = -2b$, that is, $b = \frac{2}{3}$, and $a \cdot \frac{2}{3}(-\frac{1}{3}) = -\gamma M a^{-2}$, from which follows $a = (9\gamma M/2)^{1/3}$. Thus $s(t) = a \cdot t^{2/3}$ is a solution. It is easy to check that any function of the form

$$s(t) = a(c \pm t)^{2/3} \quad \text{with} \quad a = (9\gamma M/2)^{1/3}, \quad c \text{ arbitrary}, \tag{8}$$

is a solution to the differential equation (7) as long as $c \pm t > 0$. Note that none of the solutions from this collection satisfies the initial conditions mentioned above. The solution

$$s(t) = a\left(\frac{R\sqrt{2R}}{\sqrt{9\gamma M}} - t\right)^{2/3},$$

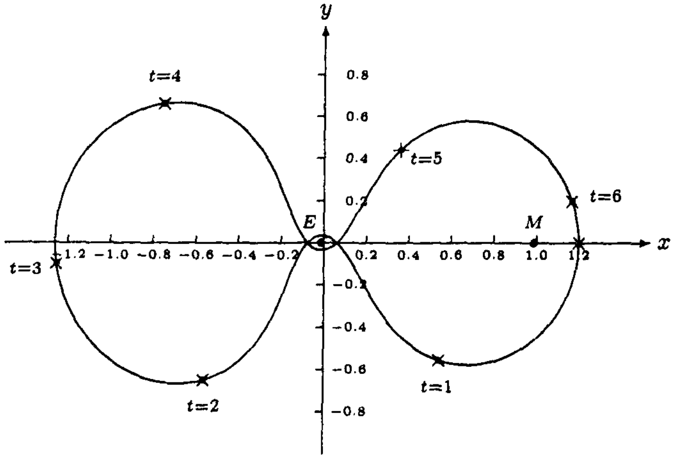
for example, satisfies $s(0) = R$, but $v(0) = \dot{s}(0) = -\sqrt{2\gamma M/R}$. This describes an object falling to earth from the position $s = R$ with initial speed at time $t = 0$ equal to $\sqrt{2\gamma M/R}$.

One of the solutions of (8) with $c = 0$ is

$$\tilde{s}(t) = at^{2/3}. \tag{9}$$

An object on this trajectory does not return to earth, since $\tilde{s}(t) \to \infty$ as $t \to \infty$; however, the velocity $\tilde{v}(t) = \frac{2}{3}at^{-1/3}$ tends to 0 as $t \to \infty$. Since $\tilde{v}(t)^2 \cdot \tilde{s}(t) =$

[4]The word *ansatz* is a German word that has become part of modern mathematical language; it has no exact English counterpart. An ansatz is an "educated guess" at the probable form of a solution. The plural of *ansatz* is *ansätze*.




$\frac{4}{9}a^3 = 2\gamma M$, the velocity as a function of distance $R$ from the earth's center can be expressed in the form

$$v_R = \sqrt{2\gamma M/R}.$$

Substituting $R = 6.370 \cdot 10^8$ cm and $M = 5.97 \cdot 10^{27}$g for the radius and mass of the earth and taking $\gamma = 6.685 \cdot 10^{-8}$ dyn $\cdot$ cm$^2 \cdot$ g$^{-2}$, one obtains

$$v_R = 1.12 \cdot 10^6 \,\text{cm/sec} = 11.2\,\text{km/sec}.$$

This is the well-known "escape velocity," the minimum velocity that a projectile fired from the surface of the earth must have in order to escape the effect of the earth's gravitational pull and never return. Compare this result with the exercise at the end of this introduction.

**III. Motion in the Gravitational Field of Two Bodies (Satellite Orbits).** The following equations (10) describe the motion of a small body (a satellite) in the force field of two larger bodies (earth and moon). It is assumed here that the motion of the three bodies takes place in a fixed plane and that the two larger bodies rotate with the same constant angular velocity about their common center of mass and maintain a constant distance to it. In particular, the effect of the small body on the motion of the two larger bodies will be ignored (this is the meaning of the adjectives 'small' and 'large'). In a corotating coordinate system with the center of mass at the origin, the two larger bodies appear to be at rest. The path of the small body can be described by a function pair $(x(t), y(t))$ that satisfies the following system of two second order differential equations:

$$\begin{aligned}\ddot{x} &= x + 2\dot{y} - \mu' \frac{x+\mu}{[(x+\mu)^2+y^2]^{3/2}} - \mu \frac{x-\mu'}{[(x-\mu')^2+y^2]^{3/2}},\\ \ddot{y} &= x - 2\dot{x} - \mu' \frac{y}{[(x+\mu)^2+y^2]^{3/2}} - \mu \frac{y}{[(x-\mu')^2+y^2]^{3/2}}.\end{aligned} \tag{10}$$

Here the two larger bodies are assumed to lie on the $x$-axis, and the parameter $\mu$, respectively $\mu'$, is the ratio of the mass of the body lying on the positive, respectively negative, $x$-axis to the combined mass of both bodies. Further, the unit of length is chosen such that the distance between the two bodies is equal to 1, and the unit of time such that the angular velocity of the rotation is also equal to 1 (i.e., a complete revolution lasts $2\pi$ time units). A closed orbit is reproduced in the figure. Here $\mu \approx 0.01213$, which corresponds to the mass ratio of the earth–moon system. The initial conditions are

$$x(0) = 1.2, \quad y(0) = 0,$$
$$\dot{x}(0) = 0, \qquad \dot{y}(0) \approx -1.04936.$$

The period $T$ (duration of one complete revolution) is approximately equal to 6.19217.

These examples suggest a variety of problems. First we made use of elementary methods of solution and discovered in the process that for some differential equations all solutions can be given in closed form (Examples I, II). For differential equations in general, just as in the problem of finding the antiderivative of an elementary function in integral calculus, the adage holds: Explicit solutions are the exception! The theory of differential equations proper has as its goal a general theory of existence, uniqueness, and other related subjects (for example, continuous dependence of solutions on various kinds of data) together with qualitative statements about the behavior of solutions in the large such as boundedness, oscillation properties, stability, and asymptotic behavior. Theorems about inequalities are also important, as the exercise at the end of this introduction illustrates.

Several important topics can only be touched briefly in an introductory work like this one. These include, for instance, the investigation of *periodic solutions* to nonlinear differential equations. Periodic solutions have important applications in mechanics (oscillations) and celestial mechanics (closed orbits). However, their mathematical theory is often difficult. Some results in this direction will be presented in 3.VI–VII and 11.X–XI. For the earth–moon–satellite problem described in III, a special case of the "restricted three-body problem," it was suggested some time ago that a spaceship on a periodic orbit could be used as a kind of "bus line" between the earth and the moon. The ensuing investigation led to the discovery of a new class of periodic orbits; see Arenstorf (1963).

Also, the problem of solving differential equations numerically will not be treated here. We note that difficult numerical problems arise in connection with space flight (determining the trajectories of spacecraft). There are efficient numerical algorithms available today that allow the determination and correction of such trajectories with sufficient accuracy and a tolerable amount of computational effort; see, for instance, the work of Bulirsch and Stoer (1966), from which the algorithm that produced the above figure is taken.

**IV. Exercise.** Prove the assertion at the end of Example II. More precisely, show: If $s(t)$ is a positive solution of the differential equation (7) in the