Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x)e^{-2\pi i x \gamma}dx$$

Akram Aldroubi, Carlos Cabrelli
Stephane Jaffard, Ursula Molter
Editors

New Trends in Applied Harmonic Analysis

Sparse Representations, Compressed Sensing, and Multifractal Analysis

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Editors
Akram Aldroubi
Department of Mathematics
Vanderbilt University
Nashville, TN, USA

Carlos Cabrelli
Depto. de Matemática
Univ. de Buenos Aires
IMAS - UBA/CONICET
Buenos Aires, Argentina

Stephane Jaffard
UFR de Sciences et Technologie
Universite Paris Est Creteil
Creteil Cedex, Paris, France

Ursula Molter
Depto. de Matemática
Univ. de Buenos Aires
IMAS, UBA/CONICET
Buenos Aires, Argentina

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. *We* intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in applicable topics such as the following, where harmonic analysis plays a substantial role:

Biomathematics, bioengineering, and biomedical signal processing;
Communications and RADAR;
Compressive sensing (sampling) and sparse representations;
Data science, data mining, and dimension reduction;
Fast algorithms;
Frame theory and noise reduction;
Image processing and super-resolution;
Machine learning;
Phaseless reconstruction;
Quantum informatics;
Remote sensing;
Sampling theory;
Spectral estimation;
Time-frequency and Time-scale analysis—Gabor theory and Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD, USA John J. Benedetto

Foreword

The CIMPA13 Conference which took place in August 5–16, 2013, in Mar de Plata, Argentina, was entitled **New Trends in Applied Harmonic Analysis Sparse Representations, Compressed Sensing and Multifractal Analysis**. The event took place in a friendly atmosphere, encouraging interaction between speakers and participants, among them PhD students, postdocs, and senior scientists. Unfortunately not all the main speakers have been able to provide a written version of their presentation, but in many cases one may find slides of more formal talks through the Internet. General information about the conference can be found at

`http://www.nuhag.eu/cimpa13`

The topics of the articles which appear in this volume reflect the diversity of recent developments in harmonic analysis, both at the level of pure mathematics and applications. Some contributions concern interesting mathematical questions arising from a systematic investigation of structures which have not been sufficiently well explored so far, and others – such as sparsity with respect to non-orthogonal systems – are part of a current trend, related to compressed sensing.

To be more precise, let us take a look at the individual contributions: The first three chapters describe problems related to multifractal analysis (Kathryn E. Hare, Stephane Seuret, and Yanick Heurteaux).

We then find two chapters thematizing the sparsity of wavelet coefficients. In the first contribution (by Vladimir Temlyakov), Lebesgue-type inequalities for greedy approximations are discussed, demonstrating that many of the well-known expansions have the following nice property: Given the set of, say, wavelet coefficients of a given function in some Besov space (because these spaces can be characterized by weighted summability conditions with respect to a given wavelet system), it is a good strategy (not only in the Hilbert spaces setting) to just take more and more of the "large coefficients" in order to approximate the function, in fact with an optimal rate.

In the second chapter in this direction, written by Eugenio Hernandez and María de Natividade, we learn some *results on nonlinear approximation for wavelet bases in weighted function spaces*. Here Bernstein- and Jackson-type theorems for weighted L^p-spaces are provided, showing that wavelet expansions are doing a good job for the approximation of functions in this setting.

The chapter provided by Pete Casazza and Janet C. Tremain discusses *the consequences of the Marcus/Spielman/Srivasta solution to the Kadison-Singer problem* in the context of frame theory with some first glimpse on the consequences within harmonic analysis.

The chapter "Model Sets and New Versions of Shannon's Sampling Theorem" by Basarab Matei presents some interesting insight on universal sampling sets, the so-called model sets and their relations to quasicrystals. While the classical Shannon theorem describes how one can recover a band-limited signal, given the *spectral support* Ω (the support of $\hat{f}$), with a formula which obviously depends on the choice of this set, the new approach discusses situations where the same sampling set can be used (with a more complicated recovery algorithm) for a large variety of sets Ω, as long as their measure is not too big.

The section written by Xianfeng Hu, Yang Wang, and Qiang Wu treats a somewhat unusual and therefore very interesting topic: *Stylometry and Mathematical Study of Authorship*.

The final contribution, entitled "Thoughts on Numerical and Conceptual Harmonic Analysis," provided by the author of this introduction gives a glimpse on a problem within the community of harmonic analysts which should be given a bit more attention: the interaction between principles of abstract (or as he proposes conceptual harmonic analysis) and those who are involved in numerical resp. computational harmonic analysis. While the first group is searching for general structures, the second one is looking for efficient algorithms and their implementation, often using FFT-based algorithms. The aspect lost in this separation of duties is the connection between the two approaches, the question, which function spaces are suitable to describe the errors made by moving from the continuous, to the discrete, and then of course to the finite setting. The article is just providing a few thoughts in this direction and suggests to pay more attention to it, not just in the spirit of function spaces or pure functional analysis but more in the sense of constructive approximation theory, with quantitative error bounds, estimates for the required problem size if one needs a guaranteed estimate for the size of the error.

Thus in some sense the article describes the ideas and goals behind the material presented by the author during the conference in a more concrete but less reflected format. Important parts of those presentations are available in the form of PDF files from www.nuhag.eu.

Overall it is clear from this volume that harmonic analysis at large is and will provide a wide variety of interesting mathematical problems and that research in this direction will continue to be fruitful and rewarding for those interested in mathematical analysis in general, be it abstract or more application oriented.

Vienna, Austria — Hans Feichtinger
October 2015

Preface

This book evolved from the written notes that were distributed to the students who participated in the CIMPA school, *New Trends in Applied Harmonic Analysis: Sparse Representations, Compressed Sensing and Multifractal Analysis*, which took place in Mar del Plata (Argentina) in August 2013.

This event was motivated by the recent interactions which developed between harmonic analysis and signal and image processing during the last 10 years. During that time, several technological deadlocks were solved through the resolution of deep theoretical problems in harmonic analysis. The purpose of this school was to focus on two particularly active areas which are representative of such advances: multifractal analysis and compressed sensing. The courses were taught by leaders in these areas and covered both theoretical aspects and applications. Most of the attendance was composed of PhD students and postdocs from diverse backgrounds (mathematics, signal and image processing, etc.), and the corresponding chapters of this book reflect the pedagogical care of the lecturers, in particular in the careful treatment of all needed prerequisites, and the illustration of the developments of each topic by several examples. Another original feature of this book is that some subjects overlap, with views taken from different perspectives, thus offering an in-depth picture of these scientific areas.

Let us be more specific. Multifractal analysis offers new tools of classification for signals and images derived from their scaling invariance properties. The part of the book concerning this subject include the contribution of K. Hare, "Multifractal Analysis of Cantor-like Measures," which deals with basics of fractal analysis and then focuses on the key example of Cantor-like measures. The contribution of Y. Heurteaux "An introduction to Mandelbrot cascades" goes one step further in modeling complexity and deals with the multifractal measures supplied by multiplicative cascades; a careful treatment of these examples is motivated both by the historical role played by these measures as models for the dissipation of energy in turbulent fluids and by the importance that they have recently acquired in other areas of mathematics (fragmentation, coalescence, harmonic measure associated with fractal sets, Schramm-Loewner evolution, etc.). Finally, the contribution of Stéphane Seuret "Multifractal analysis and Wavelets" deals with the extensions that these ideas have known in the setting of functions. The main tool here is wavelet analysis, a tool which is now prevalent in applied analysis and reappears in several other chapters of this book. Here its role is to yield a characterization of both pointwise and global regularity of functions. This property explains the success of wavelets in applied multifractal analysis, since this subject can be seen as unfolding the relationships between pointwise and global regularity and then deriving practical classification tools from these regularity characteristics.

Recently, many powerful techniques have been developed emphasizing the role of sparsity in signal and image processing. These new methods have had a substantial impact in areas like sampling, data compression and representation, atomic decompositions, wavelets, frames, and high-dimensional data analysis. In particular compressed sensing represents a new paradigm in signal and image processing, allowing to reconstruct compressible data from the knowledge of an underdetermined system, through an ℓ^1 minimization. The mathematics behind these methods is rich and sophisticated and presents new challenges. The chapters by Temlyakov "Lebesgue-type Inequalities for Greedy Approximation" and Hernández et. al "Results on Nonlinear Approximation for Wavelet Bases in Weighted Function Spaces" are excellent examples of the advances in this area.

On another note, just before the school took place, the *Kadison-Singer conjecture* was solved, and since this had deep impact on harmonic analysis – because of the implications with respect to the decomposition of frames into a finite number of Riesz bases *Feichtinger conjecture* – Pete Casazza gave a really nice lecture about the diverse attempts in the solution and agreed to write a chapter about all the implications.

Note that the contribution of Y. Heurteaux was not part of the courses taught at the CIMPA school of August 2013, but grew from the notes of another course taught at a fractal conference that took place in Porquerolles (France) in September 2013.

Nashville, TN, USA — Akram Aldroubi
Buenos Aires, Argentina — Carlos Cabrelli
Paris, France — Stephane Jaffard
Buenos Aires, Argentina — Ursula Molter
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- IMU, International Mathematical Union

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