Principles of Naval Architecture

Volume III • Motions in Waves and Controllability

THE SOCIETY OF NAVAL ARCHITECTS & MARINE ENGINEERS

Principles of Naval Architecture Second Revision

Volume III • Motions in Waves and Controllability

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Preface

The aim of this second revision (third edition) of the Society's successful *Principles of Naval Architecture* was to bring the subject matter up-to-date through revising or rewriting areas of greatest recent technical advances, which meant that some chapters would require many more changes than others. The basic objective of the book, however, remained unchanged: to provide a timely survey of the basic principles in the field of naval architecture for the use of both students and active professionals, making clear that research and engineering are continuing in almost all branches of the subject. References to available sources of additional details and to ongoing work to be followed in the future are included.

The preparation of this third edition was simplified by an earlier decision to incorporate a number of sections into the companion SNAME publication, *Ship Design and Construction,* which was revised in 1980. The topics of Load Lines, Tonnage Admeasurement and Launching seemed to be more appropriate for the latter book, and so Chapters V, VI, and XI became IV, V and XVII respectively, in *Ship Design and Construction.* This left eight chapters, instead of 11, for the revised *Principles of Naval Architecture,* which has since become nine in three volumes.

At the outset of work on the revision, the Control Committee decided that the increasing importance of high-speed computers demanded that their use be discussed in the individual chapters instead of in a separate appendix as before. It was also decided that throughout the book more attention should be given to the rapidly developing advanced marine vehicles.

In regard to units of measure, it was decided that the basic policy would be to use the International System of Units (S.I.). Since this is a transition period, conventional U.S. (or "English") units would be given in parentheses, where practical, throughout the book. This follows the practice adopted for the Society's companion volume, *Ship Design and Construction.* The U.S. Metric Conversion Act of 1975 (P.L. 94-168) declared a national policy of increasing the use of metric systems of measurement and established the U.S. Metric Board to coordinate voluntary conversion to S.I. The Maritime Administration, assisted by a SNAME ad hoc task group, developed a *Metric Practice Guide* to "help obtain uniform metric practice in the marine industry," and this guide was used here as a basic reference. Following this guide, ship displacement in metric tons (1000 kg) represents mass rather than weight. (In this book the familiar symbol, Δ, is reserved for the displacement mass). When forces are considered, the corresponding unit is the kilonewton (kN), which applies, for example, to resistance and to displacement weight (symbol W, where $W = \rho \Delta g$) or to buoyancy forces. When conventional or English units are used, displacement weight is in the familiar long ton unit

(Continued)

(2240 lb), which numerically is 1.015 × metric ton. Power is usually in kilowatts (1 kW = 1.34 hp). A conversion table also is included in the Nomenclature at the end of each volume

The first volume of the third edition of *Principles of Naval Architecture,* comprising Chapters I through IV, deals with the essentially static principles of naval architecture, leaving dynamic aspects to the remaining volumes. The second volume consists of Chapters V Resistance, VI Propulsion and VII Vibration, each of which has been extensively revised or rewritten.

Volume III contains the two final chapters, VIII Motions in Waves and IX Controllability. Because of important recent theoretical and experimental developments in these fields, it was necessary to rewrite most of both chapters and to add much new material. But the state-of-the-art continues to advance, and so extensive references to continuing work are included.

November 1989 Edward V. Lewis
Editor

Acknowledgments

In this Volume III, the Editor wishes to thank the authors of Chapter VIII, Robert F. Beck, John F. Dalzell, Philip Mandel and William C. Webster, for stepping in to complete the chapter on Motions in Waves after the untimely death of William E. Cummins. He also acknowledges the valuable assistance of SNAME T&R Panel H-7 (Seakeeping Characteristics) chaired by David D. Moran, in reviewing and commenting on early drafts of the chapter manuscript. Generous permission was granted by D. C. Murdey and his associates in the National Research Council of Canada for us to publish excerpts from their valuable reports on a series of model tests in calm water and in waves. Drafting services were provided by Keith L. MacPhee.

The Editor also wishes to express his appreciation for John Nachtsheim's valuable efforts in guiding the completion of Chapter IX on Controllability, and to Alexander Landsburg for joining in to assist the original two authors. All three authors wish to acknowledge their indebtedness to Philip Mandel, the author of the corresponding chapter in the preceding edition. Extensive use has been made of the original text and figures. The authors also wish to thank the members of Panel H-10 (Ship Controllability) who provided useful comments, especially Abraham Taplin. Completion of this chapter was greatly facilitated by Roderick A. Barr, who assisted in organizing the chapter in its early stages, by the excellent technical review and suggestions given by John A. Youngquist, and by the drafting services of Robinson de la Cruz.

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John J. Nachtsheim, Chairman
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The Editor wishes to thank all of the Volume III authors for their fine work and full cooperation in making suggested revisions. Finally, he acknowledges the indispensable efforts of Trevor Lewis-Jones in doing detailed editing and preparing text and figures in proper format for publication.

November 1989

E. V. Lewis
Editor

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* Now retired
** Deceased
Note: The office affiliations given are those at the time of writing the chapters.

CHAPTER VIII

Robert F. Beck, William E. Cummins
John F. Dalzell, Philip Mandel
William C. Webster

Motions in Waves

Section 1
Introduction[1]

1.1 Ship motions at sea have always been a problem for the naval architect. His or her responsibility has been to insure not only that the ship can safely ride out the roughest storms but that it can proceed on course under severe conditions with a minimum of delay, or carry out other specific missions successfully. However, the problem has changed through the years. Sailing vessels followed the prevailing winds—Columbus sailed west on the northeast trades and rode the prevailing westerlies farther north on his return voyages. The early clipper ships and the later grain racers from Australia to Europe made wide detours to take advantage of the trade winds. In so doing they made good time in spite of the extra distance travelled, but the important fact for the present purpose is that they seldom encountered head seas.

With the advent of steam, for the first time in the history of navigation, ships were able to move directly to windward. Hence, shipping water in heavy weather caused damage to superstructures, deck fittings and hatches to increase, and structural bottom damage near the bow appeared as a result of slamming. Structural improvements and easing of bottom lines forward relieved the latter situation, and for many years moderately powered cargo ships could use full engine power in almost any weather, even though speed was reduced by wind and sea. The same is true even today for giant, comparatively low-powered tankers and many dry-bulk carriers.

For many years the pilot charts issued by the U.S. Navy Oceanographic Office still showed special routes for "low-powered steamers" to avoid head winds and seas. It should be emphasized that the routes shown for the North Atlantic, for example, did not involve avoiding bad weather as such, for eastbound the routes for low and high-powered steamers were the same; but they did attempt to avoid the prevailing head winds and head seas westbound that greatly reduced the speed of low-powered ships.

The situation is different for today's modern fast passenger vessels and high-powered cargo ships. In really rough head seas, their available power is excessive and must be reduced voluntarily to avoid shipping of water forward or incurring structural damage to the bottom from slamming. Hence, maintaining schedule now depends as much on ship motions as on available power.

Similarly, high-powered naval vessels must often slow down in rough seas in order to reduce the motions that affect the performance of their particular mission or function—such as sonar search, landing of aircraft or helicopters and convoy escort duty. Furthermore, new and unusual high-performance craft—comparatively small in size—have appeared whose performance is even more drastically affected by ocean waves. These include high-speed planing craft, hydrofoil boats, catamarans and surface effect ships, most but not all being developed or considered for military uses.

A very different but related set of problems has arisen in the development of large floating structures and platforms that must be towed long distances and be accurately positioned in stormy seas for ocean-drilling and other purposes.

As seakeeping problems have thus became more serious, particularly for the design of higher-speed oceangoing vessels, rapid expansion began in the mid-1950s in the application of hydrodynamic theory, use of experimental model techniques and collection of full-scale empirical data. These important developments led to a better understanding of the problems and ways of dealing with them. Along with remarkable advances in oceanography and computer technology, they made it possible to predict in statistical terms many aspects of ship performance at sea. Furthermore, they could be applied to the seagoing problems involved in the design of the unusual new high-speed craft and floating platforms previously mentioned.

[1] This section written by the editor.

In view of the increasing importance of theoretical approaches to seakeeping problems, it is felt to be essential to cover in this chapter in a general way the basic hydrodynamic principles and mathematical techniques involved in predicting ship motions in both regular and irregular seas (Sections 2, 3 and 4). Some readers may wish to proceed directly to Sections 5-8, which discuss more practical aspects of ship motions and the problems of design for good seakeeping performance.

The understanding of ship motions at sea, and the ability to predict the behavior of any ship or marine structure in the design stage, begins with the study of the nature of the ocean waves that constitute the environment of the seagoing vessel. The outstanding characteristic of the open ocean is its irregularity, not only when storm winds are blowing but even under relatively calm conditions. Oceanographers have found that irregular seas can be described by statistical mathematics on the basis of the assumption that a large number of regular waves having different lengths, directions, and amplitudes are linearly superimposed. This powerful concept is discussed in Section 2 of this chapter, but it is important to understand that the characteristics of idealized regular waves, found in reality only in the laboratory, are also fundamental for the description and understanding of realistic irregular seas.

Consequently, in Section 2—after a brief discussion of the origin and propagation of ocean waves—the theory of regular gravity waves of simple form is presented. Mathematical models describing the complex irregular patterns actually observed at sea and encountered by a moving ship are then discussed in some detail. The essential feature of these models is the concept of a *spectrum,* defining the distribution of energy among the different hypothetical regular components having various frequencies (wave lengths) and directions. It is shown that various statistical characteristics of any seaway can be determined from such spectra. Sources of data on wave characteristics and spectra for various oceans of the world are presented.

It has been found that the irregular motions of a ship in a seaway can be described as the linear superposition of the responses of the ship to all the wave components of such a seaway. This principle of superposition, which was first applied to ships by St. Denis and Pierson (1953),[2] requires knowledge of both the sea components and the ship responses to them.

Hence, the vitally important linear theory of ship motions in simple, regular waves is next developed in Section 3. It begins with the simple case of pitch, heave and surge in head seas and then goes on to the general case of six degrees of freedom. The *equations of motion* are presented and the hydrodynamic forces evaluated on the basis of potential theory. The use of *strip theory* is then described as a convenient way to perform the integration for a slender body such as a ship.

Finally, practical data and experimental results for two cases are presented: the longitudinal motions of pitch-heave-surge alone, and the transverse motions of roll-sway-yaw.

In Section 4 the extension of the problem of ship motions to realistic irregular seas is considered in detail, the object being to show how modern techniques make it possible to predict motions of almost any type of craft or floating structure in any seaway in probability terms. It is shown that, knowing the wave spectrum and the characteristic response of a ship to the component waves of the irregular sea, a response spectrum can be determined. From it various statistical parameters of response can be obtained, just as wave characteristics are obtainable from wave spectra. Responses to long-crested seas are treated first, and then the more general case of short-crested seas. Particular attention is given to the short-term statistics of peaks, or maxima, of responses such as pitch, heave and roll; both motions and accelerations. Examples of typical calculations are included.

Section 5 considers the prediction of responses other than the simple motions of pitch, heave, roll, etc. These so-called *derived responses* include first the vertical motion (and velocity and acceleration) of any point in a ship as the result of the combined effect of all six modes, or degrees of freedom.

Consideration is given next to the relative motion of points in the ship and the water surface, which leads to methods of calculating probabilities of shipping water on deck, bow emergence and slamming. Non-linear effects come in here and are discussed, along with non-linear responses such as added resistance and power in waves. Finally, various wave-induced loads on a ship's hull structure are considered, some of which also involve non-linear effects.

Section 6 discusses the control of ship motions by means of various devices. Passive devices that do not require power or controls comprise bilge keels, anti-rolling tanks and moving weights. Five performance criteria for such devices are presented, and the influence of each is shown by calculations for a ship rolling in beam seas. Active devices, such as gyroscopes, controllable fins and controlled rudders are then discussed.

Section 7 deals with criteria and indexes of seakeeping performance. It is recognized that, in order for new designs to be evaluated and their acceptability determined, it is essential to establish standards of performance, just as in other chapters where criteria of stability, subdivision and strength are presented.

Various desirable features of ship behavior have been listed from time to time under the heading of *seakindliness.* These include easy motions, (i.e., low

[2]Complete references are listed at end of chapter.

accelerations), dry decks, absence of slamming and propeller racing, and easy steering. For naval ships important additional considerations include weapon system performance, landing of helicopters and sonar search effectiveness. This section considers in detail specific criteria by which to judge whether or not a ship can carry out a particular mission in a given sea condition, speed and heading. These criteria usually involve values of motion amplitude, velocity or acceleration at specific locations in the ship, or motions relative to the sea affecting shipping of water and slamming. Available prescribed values of acceptable performance are tabulated for different types of craft and various missions.

However, whether or not a ship can meet any of the criteria depends on factors such as sea condition, speed and heading. Therefore, a *Seakeeping Performance Index* (SPI) is needed that takes account of all the different sea conditions expected over a period of time and the speeds and headings attainable in each. It should measure the effectiveness of a ship in attaining its mission or missions in service. Two basic SPIs are described: A *Transit Speed* SPI and a *Mission Effectiveness* SPI. The first applies particularly to merchant ships whose mission is to deliver cargo and passengers safely and promptly, and is expressed as attainable average speed over one or more voyages without exceeding the applicable criteria. This SPI also applies to some functions of naval ships. The second SPI, Mission Effectiveness, applies particularly to naval vessels, but also to Coast Guard cutters, fishing vessels on fishing grounds, oceanographic ships and floating platforms. For such ships the SPI defines the effectiveness of the ship in fulfilling specific missions or functions, usually in terms of the fraction of time that the ship can do so over a stated period. Methods of calculating these SPIs are given, along with specific examples.

Finally, having criteria and indexes of performance whereby predictions can be tested, the naval architect requires guidance as to choice of ship form, proportions, natural periods of rolling and pitching, freeboard forward and other characteristics favorable to good seagoing performance. In Section 8 the theoretical principles and experimental data developed in preceding sections are applied to providing such needed guidelines. Emphasis is on choosing the overall ship proportions and coefficients, since they must be established early in the design process and are shown to have more influence on performance than minor changes in full form. Consideration is also given to above-water form and freeboard, and to added power requirements in waves. Special design problems of high-performance craft are discussed.

Consideration is also given to design procedures that permit seakeeping considerations to be taken into account from the outset. It is shown that a choice among alternative designs can be made on the basis of economic considerations, for both commercial and naval vessels.

Section 2
Ocean Waves[3]

2.1 Origin and Propagation of Ocean Waves. As noted in Section 1, the outstanding visible characteristic of waves in the open ocean is their irregularity. Study of wave records confirms this irregularity of the sea, both in time and space. However, one is equally impressed by the fact that over a fairly wide area and often for a period of a half-hour or more the sea may maintain a characteristic appearance, because record analyses indicate it is very nearly statistically steady or *stationary*. At other times or places the sea condition will be quite different, and yet there will again be a characteristic appearance, with different but steady statistical parameters. Hence, for most problems of behavior of ships and floating structures at sea, attention can be focused on describing mathematically the surface waves as a random, or *stochastic*, process under short-term statistically stationary conditions. Analysis of wave records has also shown that under such conditions they are approximately Gaussian in character, i.e., wave elevations read at random or at regular intervals of time have roughly a Gaussian, or normal, probability density function. This characteristic greatly simplifies the application of statistics, probability theory and Fourier analysis techniques to the development of suitable models.

The theory of seakeeping uses such mathematical models of ocean waves, which account for variability of waves in time and space, so long as conditions remain steady, permitting estimates of short-term ship performance for realistic environmental conditions over a relatively small area. These theories are based upon mathematical wave theory as well as on the laws of probability and statistics. The details of one model, particularly as they concern the naval architect, will be developed in this section.

However, for an overall understanding, as well as for solving some seakeeping problems, the variation in waves over long periods of time and over great distances cannot be overlooked. It is useful, therefore,

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[3] By William E. Cummins, with paragraphs by John F. Dalzell.

to review the physical processes of storm wave generation and of wave propagation in a general way.

Storm waves are generated by the interaction of wind and the water surface. There are at least two physical processes involved, these being the friction between air and water and the local pressure fields associated with the wind blowing over the wave surface. Although a great deal of work has been done on the theory of wave generation by wind, as summarized by Korvin-Kroukovsky (1961) and Ursell (1956), no completely satisfactory mechanism has yet been devised to explain the transfer of energy from wind to sea. Nevertheless, it seems reasonable to assume that the total storm wave system is the result of many local interactions distributed over space and time. These events can be expected to be independent unless they are very close in both space and time. Each event will add a small local disturbance to the existing wave system.

Within the storm area, there will be wave interactions and wave-breaking processes that will affect and limit the growth and propagation of waves from the many local disturbances. Nevertheless, wave studies show that if wave amplitudes are small the principle of linear superposition governs the propagation and dispersion of the wave systems outside the generating area. Specifically, if $\zeta_1(x, y, t)$ and $\zeta_2(x, y, t)$ are two wave systems, $\zeta_1(x, y, t) + \zeta_2(x, y, t)$ is also a wave system. This implies that one wave system can move through another wave system without modification. While this statement is not absolutely true, it is very nearly so, except when the sum is steep enough for wave breaking to occur.

A second important characteristic of water waves that affects the propagation of wave systems is that in deep water the phase velocity, or *celerity*, of a simple regular wave, such as can be generated in an experimental tank, is a function of wavelength. Longer waves travel faster than shorter waves. Study and analysis of ocean wave records has shown that any local system can be resolved into a sum of component regular waves of various lengths and directions, using Fourier Integral techniques. By an extension of the principle of superposition, the subsequent behavior of the sum of these component regular wave systems will determine the visible system of waves. Since these component waves have different celerities and directions, the propagating pattern will slowly change with time.

If the propagating wave system over a short period of time is the sum of a very large number of separate random contributions, all essentially independent, the surface elevation is

$$\zeta(x, y, t) = \sum_i \zeta_i(x, y, t) \qquad (1)$$

and the laws of statistics yield some very useful conclusions. Since water is incompressible, the average value of vertical displacement at any instant, t, in a regular component wave, ζ_i, is zero (if it is assumed to be of sinusoidal form, as discussed subsequently), and therefore the average value for the wave system, $\langle\zeta(x, y, t)\rangle$, is also zero. However, the variance (or mean square deviation from the mean) of ζ_i, which is the average value of ζ_i^2, written $\langle\zeta_i^2\rangle$, is a positive quantity that measures the severity of the sea. A fundamental theorem of statistics states that the variance of the sum of a set of independent random variables tends asymptotically to the sum of the variances of the component variables. Thus, for a very large (infinite) number of components, assumed to be independent,

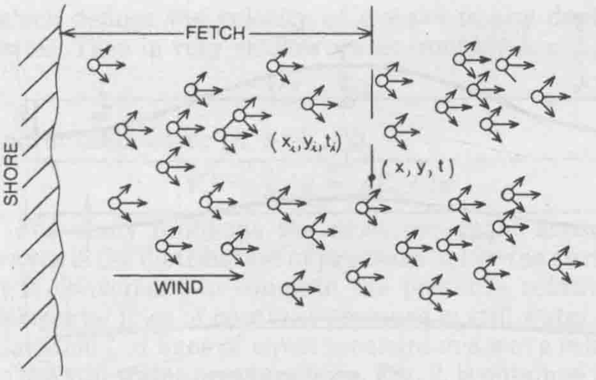
$$\langle\zeta^2\rangle = \sum \langle\zeta_i^2\rangle \qquad (2)$$

A final statistical conclusion is a consequence of the *central limit theorem* of statistics. In the case under discussion, this theorem implies that $\zeta(x, y, t)$ will have a normal (or Gaussian) density function, even if the component variables $\zeta_i(x, y, t)$ are not distributed normally. The importance of this result is that the density function of a normal random variable is known if its mean and variance are known. Therefore, if the variance of the surface elevation in the multi-component wave system can be estimated, its probability density as a random variable is known. Ochi (1986) deals with the analysis of non-Gaussian random processes.

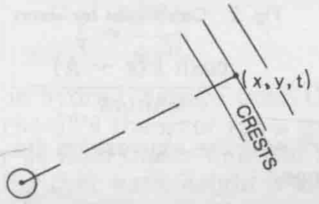
These conclusions from the laws of statistics all depend upon the previously mentioned principle of superposition, which holds approximately but not absolutely for water waves, and on the assumption of independence of component waves. Therefore, the conclusions themselves are approximate and this should be remembered. However, it has been found that over the short term, deviations become significant only when the waves are very steep, and even then primarily in those characteristics that are strongly influenced by the crests.

It will be shown that the short-term descriptive model that has been described leads to a mathematical technique for describing the irregular sea at a given location and time, while conditions remain steady or *stationary.* Each sea condition can for short periods of time be as unique as a fingerprint, and yet, as with a fingerprint, it has order and pattern, as defined by its directional *spectrum,* to be explained subsequently (Section 2.6). However, since the wind velocities and directions are continually, albeit slowly, changing, the short-term mathematical description will also change. Hence, a broader model is also needed to cover large variations in time, involving wind effects on growth and decline of local wave systems, as well as propagation and dispersion.

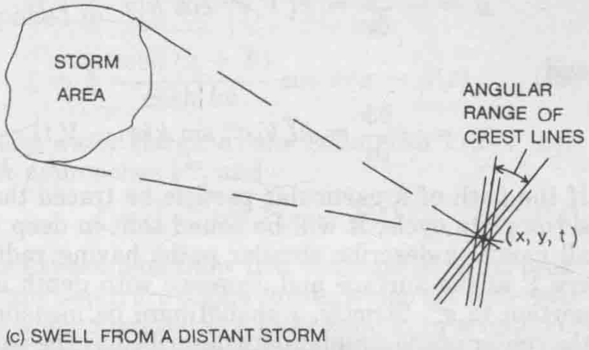
Fig. 1(a) symbolizes a storm-wave generation area. It may be assumed that disturbances are being generated by the interaction of the wind and sea surface throughout the storm area from the time the wind



(a) STORM WAVE GENERATION AREA



(b) WAVE PROPAGATION FROM A LOCAL DISTURBANCE



(c) SWELL FROM A DISTANT STORM

Fig. 1 Ocean wave generation and propagation

starts to blow over the region. Fig. 1(b) shows the effect at an observation point (x, y, t) of a disturbance at (x_i, y_i, t_i). Since a specific disturbance creates a dispersive wave system originating with a local interaction between wind and sea, it has the form of radiating waves spreading from the point (x_i, y_i). At any distant observation point it will appear to be a system of locally long-crested waves progressing from the direction of the point of origin. The original action (e.g., an impulsive displacement) is assumed to generate a band of frequencies, each corresponding to a different band of wavelengths. As different wavelengths advance at different celerities, the longest waves will reach the observation point first, and the observed average wavelength will decrease with increasing time, $(t - t_i)$. The total wave displacement, ζ, at the observation point is the sum of effects due all disturbances in the generation area that are upwind of a line through the observation point perpendicular to the wind direction. Because of angular dispersion, or *spreading,* the many wave systems will come from different directions, and the combined system will generally show *short-crestedness.*

If there is a boundary to windward of the generation area, a shore or the edge of the storm, the total wave systems at a series of observation points will differ in character as the points approach the boundary, as there will be fewer disturbances propagating over the observation point. This distance from the observation point to the boundary is called the *fetch.* Also, if the waves are observed at a fixed point, starting with the inception of the wind, the wave system will grow with time. The time interval between storm inception and observation is called *duration.* If wind speed is steady, while fetch and/or duration are increased, the sea condition eventually takes on a statistically stable structure which is called *fully developed.* Further increases in fetch and duration have no significant effect on the statistical characteristics of the wave pattern.

If the observation point is outside the storm area, Fig. 1(c), then it is seen that the arriving seas, now called *swells,* clearly have a more regular character, depending upon the distance and area of the storm. The crests of the various component wave trains become more nearly parallel as the observation point recedes from the storm area, with the result that actual waves become more and more *long-crested,* that is, the identifiable length of a wave crest becomes large compared with the spacing between crests. Distance or fetch has the effect of limiting the range of wavelengths (frequencies) reaching an observation point at a given time, i.e., the greater the distance, the narrower the bandwidth of frequencies. This filtering effect is due to the different celerities of the different component wavelengths. The lengths of waves in this band decrease with time, with the shortest identifiable components being greatly attenuated and perhaps arriving well after the storm has passed. These qualities of long-crestedness and limited bandwidth are responsible for the characteristic regular appearance of swell.

A complete long-term description can best be provided by specifying many spectra (short-term) for different points throughout the area under consideration, and at regular increments of time. Despite the lack of an entirely satisfactory theory of wave generation, oceanographers have devised semi-empirical methods of predicting the changing wave spectra by considering the effect of winds on the growth or decay of local wave systems. For example, Pierson, et al (1955) described a method of accomplishing this, making use of theoretical work of Phillips and Miles, as well as empirical data. See Section 2.9.

Since the short-term irregular wave patterns observed at sea will be described in terms of regular component wave trains of different frequency and direction, it is important to consider next the characteristics of simple gravity waves.

2.2 Theory of Simple Gravity Waves. In the hydrodynamic theory of surface waves it is assumed that the crests are straight, infinitely long, parallel and equally spaced, and that wave heights are constant. The wave form advances in a direction perpendicular to the line of crests at a uniform velocity, V_c, usually referred to as *celerity* to emphasize that it is the wave form rather than the water particles that advances. Such simple waves are usually referred to as two-dimensional waves. It is assumed in wave theory that water has zero viscosity and is incompressible. It is convenient also to assume that, although waves are created by wind forces, atmospheric pressure on the water surface is constant after the wave train has been established.

The surface wave is the visible manifestation of pressure changes and water-particle motions affecting the entire body of fluid—theoretically to its full depth. The motion of particles under the idealized conditions can be characterized conveniently by a quantity known as the *velocity potential* ϕ which is defined as a function whose negative derivative in any direction yields the velocity component of the fluid in the same direction. From this function all of the desired wave characteristics can be derived. Treatises on hydrodynamics give the velocity potential for a two-dimensional wave in any depth of water and express the resulting wave form by a Fourier series (Korvin-Kroukovsky, 1961; Lamb, 1924). If certain simplifications are introduced, which amount to assuming the waves to be of very small (theoretically infinitesimal) amplitudes, the so-called first-order theory reduces the wave to the first harmonic alone. (A more exact solution is discussed in Section 2.3). The simplified potential is as follows:

$$\phi = -\bar{\zeta}\, V_c \frac{\cosh k(z+h)}{\sinh kh} \cdot \sin k(x - V_c t) \qquad (3)$$

The origin is taken at the still-water level directly over a hollow, Fig. 2; x is the horizontal coordinate, positive in the direction of wave propagation, and z is the vertical coordinate, positive upward. This positive upward convention is adopted for consistency with the work on ship motions to follow, although it differs from some references. Also

$\bar{\zeta}$ is surface wave amplitude (half-height from crest to trough)
L_w is wave length
h is depth of water
k is the *wave number*, $2\pi / L_w$
V_c is wave velocity or *celerity*
t is time

For the case of deep water (roughly $h > L_w/2$) the ratio

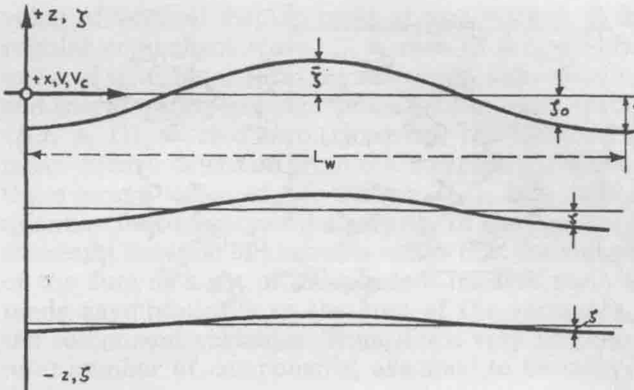


Fig. 2 Coordinates for waves

$$\frac{\cosh k(z+h)}{\sinh kh}$$

approaches e^{kz} and the expression for the velocity potential becomes

$$\phi = -\bar{\zeta}\, V_c e^{kz} \sin k(x - V_c t) \qquad (4)$$

Hence, the horizontal and vertical components of water velocity at any point in deep water are given by

$$u = -\frac{\partial \phi}{\partial x} = k\bar{\zeta}\, V_c e^{kz} \cos k(x - V_c t) \qquad (5)$$

and

$$w = -\frac{\partial \phi}{\partial z} = k\bar{\zeta}\, V_c e^{kz} \sin k(x - V_c t) \qquad (6)$$

If the path of a particular particle be traced through a complete cycle, it will be found that in deep water all particles describe circular paths having radii that are $\bar{\zeta}$ at the surface and decrease with depth in proportion to e^{kz}. Strictly, z should here be measured to the center of the circular path described by the particle. In shallow water the particles move in ellipses with a constant horizontal distance between foci and with vertical semi-axes varying with depth. At the bottom, the vertical semi-axis is zero, and the particles oscillate back and forth on straight lines.

To determine the foregoing velocities in any particular case, it is necessary to derive an expression for wave velocity V_c. Books by Milne-Thompson (1960) and Korvin-Kroukovsky (1961), show that the conditions of velocity and pressure at the surface of the wave require that

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad (7)$$

Inserting Equation (3) for the potential in (7), it can be shown that

$$V_c^2 = \frac{g}{k} \tanh kh \qquad (8)$$

which defines the velocity of a wave in any depth of water. Then in very shallow water (roughly $h < L_w/25$)

$$V_c^2 = gh \tag{9}$$

and in deep water ($h > L_w/2$),

$$V_c^2 = g/k = gL_w/2\pi \tag{10}$$

For many problems the most important aspect of waves is the distribution of pressure below the surface. It is convenient to compute the pressure relative to horizontal lines of constant pressure in still water. The elevation ζ of lines of equal pressure in a wave relative to the still-water pressure lines, Fig. 2, is obtained from the expression

$$\zeta = \frac{1}{g}\frac{\partial\phi}{\partial t} \tag{11}$$

which is derived in hydrodynamics texts (Lamb, 1924) by means of Bernoulli's theorem for a gravity force acting on a body of fluid under uniform atmospheric pressure, assuming that wave height is small (strictly speaking, infinitesimal). Then for water of any depth

$$\zeta = \frac{k\bar{\zeta}V_c^2}{g}\frac{\cosh k(z+h)}{\sinh kh}\cos k(x - V_ct) \tag{12}$$

Since from Equation (8) $kV_c^2/g = \tanh kh$, this can be simplified to

$$\zeta = \bar{\zeta}\,\frac{\cosh(z+h)}{\cosh kh}\cos k(x - V_ct) \tag{13}$$

In deep water (large h) the ratio $\cosh k(z+h)/\cosh kh$ approaches e^{kz}, and

$$\zeta = \bar{\zeta}e^{kz}\cos k(x - V_ct) \tag{14}$$

These expressions show that contours of equal pressure at any depth are cosine curves which are functions of time when observed at a fixed point x_0 or a function of distance x at a particular instant t_0. Since e^{kz} decreases as z decreases, the contours of equal pressure are attenuated with depth, approaching zero amplitude as $z \to -\infty$. These contours are the same as those generated by the orbital motions of individual particles.

To obtain the surface *wave profile*, z is taken equal to zero in Equation (13) or (14). Then

$$\zeta_0 = \bar{\zeta}\cos k(x - V_ct) \tag{15}$$

for both deep and shallow water.

A more convenient form for the equation of a simple harmonic wave can be obtained by using circular frequency $\omega = 2\pi/T_w$. The period T_w is the time required for the wave to travel one wave length, and hence the relationship between wave length and period in deep water can be derived from Equation (10).

$$T_w = \frac{L_w}{V_c} = \frac{L_w}{(gL_w/2\pi)^{1/2}} = \left(\frac{2\pi L_w}{g}\right)^{1/2} \tag{16}$$

Hence, circular frequency

$$\omega = \frac{2\pi}{T_w} = \left(\frac{2\pi}{L_w/g}\right)^{1/2} = (kg)^{1/2} = kV_c \tag{17}$$

and

$$\zeta_0 = \bar{\zeta}\cos(kx - \omega t) \tag{18}$$

When observed at a fixed point, with $x = 0$

$$\zeta_0 = \bar{\zeta}\cos(-\omega t) = \bar{\zeta}\cos\omega t$$

Alternatively, if the wave profile is studied at $t = 0$

$$\zeta_0 = \bar{\zeta}\cos kx$$

The slope of the wave surface is obtained by differentiation:

$$\frac{d\zeta_0}{dx} = k\bar{\zeta}\sin kx \tag{19}$$

The slope is maximum when $kx = \pi/2$ and $\sin kx = 1.0$. Then

$$\text{Max}\,\frac{d\zeta_0}{dx} = k\bar{\zeta} = 2\pi\frac{\bar{\zeta}}{L_w} = \frac{\pi h_w}{L_w} \tag{20}$$

where h_w is the wave height from hollow to crest. This maximum slope occurs midway between a crest and a hollow.

The contours of constant pressure that have been derived in Equations (13) and (14) also indicate the increase or decrease in pressure relative to still water at any point in terms of depth or head. Hence, to obtain the pressure p at any point we need only multiply the head by density ρg, or

$$p = \rho g(-z + \zeta)$$

In deep water, then, from Equations (11) and (14)

$$p = -\rho gz + \bar{\zeta}\rho g e^{kz}\cos(kx - \omega t)$$

As previously noted, z should be measured to the center of the circular path described by the particle at the point in question.

Evaluation of the equation for pressure in a deep-water wave under the crest, at $2\bar{\zeta}$ below the original still-water level ($z = -2\bar{\zeta}$), gives, for example

$$p = -\rho g(-2\bar{\zeta}) + \bar{\zeta}\rho g\bar{e}^{2k\bar{\zeta}} = \rho\bar{\zeta}g(2 + \bar{e}^{2k\bar{\zeta}})$$

and for $k = 0.015$, and $\bar{\zeta} = 10$, for example:

$$p = \rho\bar{\zeta}g(2.0 + 0.74) = \rho\bar{\zeta}g(2.74)$$

If the pressure were directly proportional to depth below the surface, it would be $\rho\bar{\zeta}g(3.0)$ at this point. The difference represents the so-called *Smith effect* (Smith, 1883). Similarly, under the wave hollow at a depth of $2\bar{\zeta}$ below still-water level ($z = -2\bar{\zeta}$),

$$p = \rho\bar{\zeta}g(2.0 - 0.74) = \rho\bar{\zeta}g(1.26)$$

instead of $\rho\bar{\zeta}g(1.0)$. Thus under the crest the pressures are decreased, and under the hollow the pressures are increased, by the Smith effect.

The *energy* in a train of regular waves consists of kinetic energy associated with the orbital motion of water particles and potential energy resulting from the change of water level in wave hollows and crests. The kinetic energy can be derived from the velocity potential. For one wave length L_w the kinetic energy per unit breadth of a wave of small height is given in books on hydrodynamics (Lamb, 1924; Korvin-Kroukovsky, 1961), as

$$\frac{1}{2}\int_0^{L_w} \phi \frac{\partial \phi}{\partial z}\, dx$$

This is evaluated for a simple cosine wave as

$$\frac{1}{4}\,\overline{\zeta}^2 \rho g L_w$$

The potential energy due to the elevation of water in one wave length is obtained by taking static moments about the still-water level. A unit increment of area is $\overline{\zeta}\, dx$ and the lever arm is $\zeta_0/2$. Hence, integrating, potential energy is

$$\int_0^{L_w} \rho g(\zeta_0/2)\zeta_0\, dx$$

$$= \frac{1}{2}\rho g \int_0^{L_w} \zeta_0^2\, dx$$

For a cosine wave,

$$\zeta_0 = \overline{\zeta} \cos k(x - V_c t)$$

and at $t = 0$

$$\zeta_0 = \overline{\zeta} \cos kx$$

Hence, potential energy is

$$\frac{1}{4}\,\overline{\zeta}^2 \rho g L_w$$

These derivations show that wave energy is half kinetic and half potential when averaged over a wave length. Total energy is

$$\frac{1}{2}\rho g \overline{\zeta}^2 L_w$$

Or the average energy per unit area of surface,

$$\text{Ave. unit Energy} = \frac{1}{2}\rho g \overline{\zeta}^2 \qquad (21)$$

Another useful property of waves, especially irregular waves to be discussed in Section 2.6, is the *variance*, or the mean-square value of surface elevation as a function of time. In general, the variance of a continuous function with zero mean is given by,

$$\langle \zeta(t)^2 \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \zeta^2(t)\, dt \qquad (22)$$

where the brackets ⟨ ⟩ indicate mean value of.

In the case of a simple harmonic wave, as given by Equation (18) at $x = 0$, T can be taken as the wave period, and it can be shown that

$$\langle \zeta(t)^2 \rangle = \frac{1}{2}\overline{\zeta}^2 \qquad (23)$$

That is, the variance of wave elevation of a single cycle of a sine wave is equal to one-half the square of the amplitude. This theorem is also true for a finite number of complete cycles, or in the limit as $T \to \infty$ in Equation (22).

For the work to follow, the two-dimensional regular wave can be considered to be a three-dimensional wave train with straight, infinitely long crests, i.e., a long-crested regular wave. Furthermore, with axes fixed in the earth the surface elevation of such waves traveling at any angle, μ, to the x-axis can be described by the general equation,

$$\zeta(x, y, t) = \overline{\zeta} \cos [k(x\cos \mu + y \sin \mu) - \omega t + \epsilon] \qquad (24)$$

where ϵ is a phase angle. For the case $\mu = 0$ this equation reduces to Equation (18), except for the phase angle, which is needed when more than one wave is present.

If a fixed point at the origin is considered ($x = 0$, $y = 0$), the equation becomes

$$\zeta(t) = \overline{\zeta} \cos(-\omega t + \epsilon) \qquad (25)^4$$

Wave Properties. The following is a summary of the properties of two-dimensional harmonic waves and waves of finite height in deep water (any consistent units):

Wave number	$k = 2\pi/L_w = \omega^2/g$	
Surface profile	$\zeta_0 = \overline{\zeta} \cos k(x - V_c t)$	(15)
(first approximation)	$= \overline{\zeta} \cos(kx - \omega t)$	(18)
Velocity potential	$\phi = -\overline{\zeta}\, V_c e^{kz} \sin k(x - V_c t)$	(4)
Wave celerity	$V_c = \frac{L_w}{T} = \left(\frac{gL_w}{2\pi}\right)^{1/2} = \frac{gT_w}{2\pi} = \frac{g}{\omega}$	(10)
Wave length	$L_w = 2\pi \frac{V_c^2}{g} = \frac{gT_w^2}{2\pi}$	(16)
Wave period	$T_w = (2\pi L_w/g)^{1/2}$	(16)
Maximum wave slope (first approximation)	$k\overline{\zeta} = 2\pi \frac{\overline{\zeta}}{L_w} = \frac{\pi h_w}{L_w}$	(20)
Wave energy per unit area	$\frac{1}{2}\rho g \overline{\zeta}^2$	(21)
Wave variance	$\langle \zeta^2 \rangle = \frac{1}{2}\overline{\zeta}^2$	(23)
In feet-seconds units:		
Wave celerity	$V_c = 2.26 L_w^{1/2}$	

[4] $\overline{\zeta}$ is often taken to represent the complex amplitude, in which case the imaginary part defines the phase angle and ϵ is unnecessary.

Wave length $L_w = 5.118T_w^2 = 0.196V_c^2$

Wave period $T_w = 0.442L_w^{1/2}$

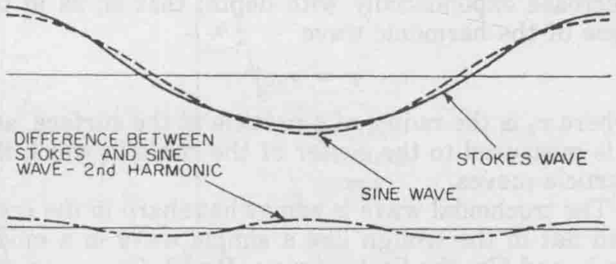


Fig. 3 Comparison of sine wave and Stokes wave

2.3 Waves of Finite Height. A hydrodynamic theory of waves of finite amplitude, i.e., not infinitesimal as previously assumed, was formulated by Stokes (1847) and others. It corresponds with the observed fact that actual waves have sharper crests and flatter hollows than the simple cosine wave assumed in the preceding section. The equation for velocity potential, which leads with approximations to the simple harmonic wave, yields a second-order wave profile when the approximations are not made. The solution can be extended with further refinements into a series expansion, and therefore, the wave form, in principle, can be expressed to any desired precision by taking a sufficient number of terms. Actually, for all practical purposes, the Stokes equation to the second order of approximation is satisfactory for ship problems. Expressed as a function of x at fixed time $t = 0$, in deep water, the surface profile is

$$\zeta_0 = \overline{\zeta} \cos kx + \pi \frac{\overline{\zeta}^2}{L_w} \cos 2kx \qquad (26)$$

In other words, the simple cosine curve is modified by a harmonic which is half the length of the fundamental, Fig. 3. The velocity of the harmonic wave, however, must be the same as for the fundamental.

As wave height increases, the crest approaches a cusp, the double angle of which is $2\pi/3$ radians or 120 deg, which corresponds to a limiting wave height from crest to trough of $0.14L_w$ or approximately $\frac{1}{7}L_w$. Real waves will break well before this height is reached.

Consideration of water-particle velocities in a wave of finite height reveals that the forward water velocity at a wave crest is greater than the backward velocity in the hollow. This difference in particle velocities, when averaged over wave length, leads to the mean velocity of water flow or *mass transport*

$$\overline{u} = k^2\overline{\zeta}^2 V_c e^{2kz} \qquad (27)$$

when z is the mean particle depth at which the velocity is sought. Hence, the particle motion is not exactly circular.

It can be seen from Equation (27) that the velocity reduces rapidly with depth. Even at the surface the drift velocity is only of the order of 2 to 3 percent of wave velocity, although it may be a significant percentage of the water-particle velocities.

While the *Stokes wave,* with its sharpened crest and flattened trough, is a more accurate geometrical model of real regular waves, it suffers from a limitation that negates its value in treating storm seas and swell, and the principle of superposition does not apply. If two Stokes waves are added, the sum is not a valid wave form. This is easily seen by simply adding two identical waves, which is equivalent to multiplying Equation (26) for ζ_0 by 2. But for this to be a valid Stokes wave, the second term should have been multiplied by 4. It has become standard practice to accept the slight errors in wave shape of linear harmonic wave theory in order to achieve simplicity in treating the additive wave systems that are characteristic of both sea and swell. Errors in form become significant when waves become steep enough to approach breaking, and when the geometry of the wave crest is a factor in the treatment of a problem. But a correct mathematical analysis of nonlinear short-crested irregular waves implies a great increase in complexity (St. Denis, 1980).

2.4 Trochoidal Waves. From the early days of naval architecture it has been customary to make use of a *trochoidal* wave in some ship-design problems. It is a convenient form from the geometrical point of view, but it fails to meet certain requirements of classical hydrodynamics and cannot be derived from the velocity potential. Its profile is almost identical with the second-order Stokes wave. In deep water all particles within trochoidal waves follow circular orbits about fixed centers at a constant angular velocity. In any horizontal line of orbit centers, the radii are equal but the phase of adjacent particles varies successively. In any vertical line, all the particles have the same phase but the radii of their orbits decrease exponentially as the depth increases. Particles which, in still water, may be identified by the intersections of a rectangular grid, take the positions shown by the intersections of the distorted grid in Fig. 4 at some instant during the passage of a wave. Those which were originally in the same horizontal line lie on undulating surfaces, while those originally in the same vertical line lie along lines which sway from side to side, converging under the crests and diverging under the hollows. The orbit centers are somewhat above the still-water positions of the corresponding particles. The wave form travels to the left when the generating circles, with fixed centers, revolve counterclockwise.

The curve joining a series of particles originally in the same horizontal plane is the same as that which is generated by a point on the radius of a circle as the circle rolls along the underside of a horizontal straight line, as is evident from a comparison of Figs. 4 and 5. Such curves, whose limit is the cycloid, are called trochoids. They are also contours of equal pressure.

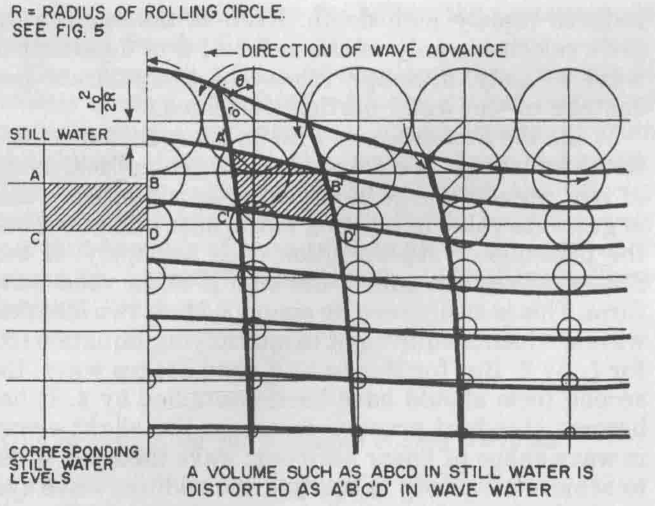


Fig. 4 Trochoidal wave motion

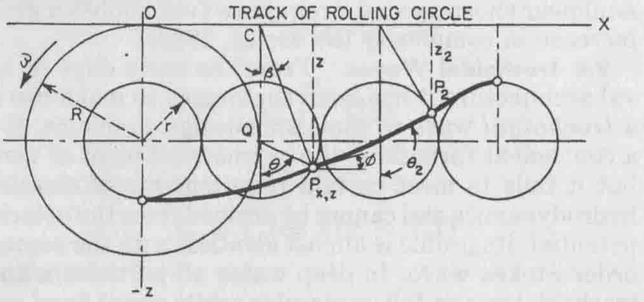


Fig. 5 Geometry of trochoid

If we call the orbit radius r and the amplitude $\overline{\zeta}$, then $\overline{\zeta} = r$. Quantities referring to the surface wave are denoted by subscript 0; thus $\overline{\zeta} = r_0$. If R is the radius of the rolling circle and L_w is the wave length from crest to crest, $L_w = 2\pi R$. To draw a trochoidal-wave surface, the selected wave length is divided by a convenient number of equally spaced points, and, with each as a center, a circle of diameter equal to the selected wave height is described. In these circles are drawn radii at successive angles which increase by the same fraction of 360 deg as the spacing of the circles in relation to wave length. The curve connecting the ends of those radii is the desired trochoid.

In Fig. 5, an ordinate z upward, and an angular velocity ω counterclockwise, are considered positive. From an initial position, shown at the left, the large circle is assumed to be rolling steadily, counterclockwise, and after time t to have reached the position OCP, having turned through the angle $\theta = \omega t$. In this case θ is positive since ω is counterclockwise.

The parametric equations of the trochoid in Fig. 5 are

$$x = R\theta + r \sin \theta = R\omega t + r \sin \omega t \qquad (28)$$
$$z = R + r \cos \theta = R + r \cos \omega t$$

The radii of the circles in which the particles move decrease exponentially with depth; that is, as in the case of the harmonic wave

$$r = r_0 e^{kz}$$

where r_0 is the radius of a particle at the surface, and z is measured to the center of the circle in which the particle moves.

The trochoidal wave is somewhat sharp in the crest and flat in the trough like a simple wave in a model tank, and like the Stokes wave, Fig. 3. Consequently, for equal water volumes the lines of orbit centers must be somewhat above the corresponding still-water levels in order that the amount of water in the crest will equal the amount removed in the hollow. It can be shown that this rise of orbit centers is $r^2/2R$, Fig. 4.

Although the trochoidal wave is reasonably realistic for waves up to about $L/20$ in height, the limiting case of $R = r$ gives an impossibly steep wave with very sharp cusps. Other characteristics of the trochoidal wave, such as velocity, period, pressure change with depth, are the same as for the simple harmonic wave previously discussed.

Obviously the pressure at any point on the surface of a wave is atmospheric. Furthermore, the sum of all the hydrodynamic and buoyant forces acting on a surface particle is perpendicular to the surface, as demonstrated by Froude with a little float carrying a pendulum. Although this statement can be proved on the basis of the theory of a simple harmonic wave, it is most easily demonstrated by means of trochoidal theory.

Following Froude's approach, it is convenient to deal with the inertial reactions to the water-pressure forces acting on the particle P in Fig. 6, although the latter could also be determined directly. As previously shown, the buoyancy and hydrodynamic pressure forces in the wave cause the particle to move in a circular path, and the equal and opposite reactions consist of gravity mg acting downward and a centrifugal reaction QF' resulting from orbital motion of the particle

$$mr\omega^2$$

It can be shown that in a trochoidal gravity wave $\omega^2 = g/R$ and therefore the centrifugal reaction is

$$mg(r/R)$$

The resultant is PF in Fig. 6.

In triangles QCP and $F'FP$, PF' is in line with QP and $F'F$ is parallel to QC. Also

$$\frac{PF'}{FF'} = \left(mg\,\frac{r}{R}\right) \Big/ mg = \frac{r}{R} = \frac{QP}{QC}$$