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To Jinghe,
À Céline,

Abstract

Using a codimension-1 algebraic cycle obtained from the Poincaré line bundle, Beauville defined the Fourier transform on the Chow groups of an abelian variety A and showed that the Fourier transform induces a decomposition of the Chow ring $\mathrm{CH}^*(A)$. By using a codimension-2 algebraic cycle representing the Beauville–Bogomolov class, we give evidence for the existence of a similar decomposition for the Chow ring of hyperKähler varieties deformation equivalent to the Hilbert scheme of length-2 subschemes on a K3 surface. We indeed establish the existence of such a decomposition for the Hilbert scheme of length-2 subschemes on a K3 surface and for the variety of lines on a very general cubic fourfold.

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Introduction

A. Abelian varieties

Let A be an abelian variety of dimension d over a field k. Let $\hat{A} = \mathrm{Pic}^0(A)$ be its dual and let L be the Poincaré line bundle on $A \times \hat{A}$ viewed as an element of $\mathrm{CH}^1(A \times \hat{A})$. The *Fourier transform* on the Chow groups with rational coefficients is defined as

$$\mathcal{F}(\sigma) := p_{2,*}(e^L \cdot p_1^*\sigma), \quad \text{for all } \sigma \in \mathrm{CH}^i(A).$$

Here, $e^L := [A \times \hat{A}] + L + \frac{L^2}{2!} + \ldots + \frac{L^{2d}}{(2d)!}$, and $p_1 : A \times \hat{A} \to A$ and $p_2 : A \times \hat{A} \to \hat{A}$ are the two projections. The main result of [**8**] is the following.

THEOREM (Beauville). *Let A be an abelian variety of dimension d. The Fourier transform induces a canonical splitting*

(1) $$\mathrm{CH}^i(A) = \bigoplus_{s=i-d}^{i} \mathrm{CH}^i(A)_s,$$

$$\textit{where } \mathrm{CH}^i(A)_s := \{\sigma \in \mathrm{CH}^i(A) : \mathcal{F}(\sigma) \in \mathrm{CH}^{d-i+s}(\hat{A})\}.$$

Furthermore, this decomposition enjoys the following two properties :

(a) $\mathrm{CH}^i(A)_s = \{\sigma \in \mathrm{CH}^i(A) : [n]^*\sigma = n^{2i-s}\sigma\}$, *where $[n] : A \to A$ is the multiplication-by-n map ;*

(b) $\mathrm{CH}^i(A)_s \cdot \mathrm{CH}^j(A)_r \subseteq \mathrm{CH}^{i+j}(A)_{r+s}$. □

Property (a) shows that the Fourier decomposition (1) is canonical, while Property (b) shows that the Fourier decomposition is compatible with the ring structure on $\mathrm{CH}^*(A)$ given by intersection product. It should be mentioned that, as explained in [**8**], (1) is expected to be the splitting of a Bloch–Beilinson type filtration on $\mathrm{CH}^*(A)$. By [**21**], this splitting is in fact induced by a Chow–Künneth decomposition of the diagonal and it is of Bloch–Beilinson type if it satisfies the following two properties :

(B) $\mathrm{CH}^i(A)_s = 0$ for all $s < 0$;

(D) the cycle class map $\mathrm{CH}^i(A)_0 \to \mathrm{H}^{2i}(A, \mathbb{Q})$ is injective for all i.

Actually, if Property (D) is true for all abelian varieties, then Property (B) is true for all abelian varieties ; see [**52**].

A direct consequence of the Fourier decomposition on the Chow ring of an abelian variety is the following. First note that $\mathrm{CH}^d(A)_0 = \langle [0] \rangle$, where $[0]$ is the class of the identity element $0 \in A$. Let $D_1, \ldots, D_d$ be symmetric divisors, that is, divisors such that $[-1]^*D_i = D_i$, or equivalently $D_i \in \mathrm{CH}^1(A)_0$, for all i. Then

(2) $$D_1 \cdot D_2 \cdot \ldots \cdot D_d = \deg(D_1 \cdot D_2 \cdot \ldots \cdot D_d)\,[0] \quad \text{in } \mathrm{CH}^d(A).$$

So far, for lack of a Bloch–Beilinson type filtration on the Chow groups of hyperKähler varieties, it is this consequence (2), and variants thereof, of the canonical splitting of the Chow ring of an abelian variety that have been tested for certain types of hyperKähler varieties. Property (2) for divisors on hyperKähler varieties F is Beauville's weak splitting conjecture [**9**] ; it was subsequently strengthened in [**56**] to include the Chern classes of F. The goal of this manuscript is to show that a Fourier decomposition should exist on the Chow ring of hyperKähler fourfolds which are deformation equivalent to the Hilbert scheme of length-2 subschemes on a K3 surface. Before making this more precise, we first consider the case of K3 surfaces.

B. K3 surfaces

Let S be a complex projective K3 surface. In that case, a Bloch–Beilinson type filtration on $\mathrm{CH}^*(S)$ is explicit : we have $\mathrm{F}^1\mathrm{CH}^2(S) = \mathrm{F}^2\mathrm{CH}^2(S) = \mathrm{CH}^2(S)_{\mathrm{hom}} := \ker\{cl : \mathrm{CH}^2(S) \to \mathrm{H}^4(S,\mathbb{Q})\}$ and $\mathrm{F}^1\mathrm{CH}^1(S) = 0$. Beauville and Voisin [**11**] observed that this filtration splits canonically by showing the existence of a zero-cycle $\mathfrak{o}_S \in \mathrm{CH}^2(S)$, which is the class of any point lying on a rational curve on S, such that the intersection of any two divisors on S is proportional to $\mathfrak{o}_S$. Let us introduce the Chow–Künneth decomposition

(3) $\pi^0_S := \mathfrak{o}_S \times S, \quad \pi^4_S := S \times \mathfrak{o}_S \quad \text{and} \quad \pi^2_S := \Delta_S - \mathfrak{o}_S \times S - S \times \mathfrak{o}_S \quad \text{in } \mathrm{CH}^2(S\times S).$

Cohomologically, π^0_S, π^2_S and π^4_S are the Künneth projectors on $\mathrm{H}^0(S,\mathbb{Q})$, $\mathrm{H}^2(S,\mathbb{Q})$, and $\mathrm{H}^4(S,\mathbb{Q})$, respectively. The result of Beauville–Voisin shows that among the Chow–Künneth decompositions of Δ_S (note that a pair of zero-cycles of degree 1 induces a Chow–Künneth decomposition for S and two distinct pairs induce distinct such decompositions) the symmetric one associated to $\mathfrak{o}_S$, denoted $\Delta_S = \pi^0_S + \pi^2_S + \pi^4_S$, is special because the decomposition it induces on the Chow groups of S is compatible with the ring structure on $\mathrm{CH}^*(S)$.

A key property satisfied by $\mathfrak{o}_S$, proved in [**11**], is that $c_2(S) = 24\,\mathfrak{o}_S$. Let $\iota_\Delta : S \to S \times S$ be the diagonal embedding. Having in mind that the top Chern class of the tangent bundle $c_2(S)$ is equal to $\iota_\Delta^*\Delta_S$, we see that the self-intersection of the Chow–Künneth projector π^2_S satisfies

$$\begin{aligned}\pi^2_S \cdot \pi^2_S &= \Delta_S^{\cdot 2} - 2\,\Delta_S \cdot (\mathfrak{o}_S \times S + S \times \mathfrak{o}_S) + (\mathfrak{o}_S \times S + S \times \mathfrak{o}_S)^{\cdot 2}\\ &= (\iota_\Delta)_* c_2(S) - 2\,\mathfrak{o}_S \times \mathfrak{o}_S\\ &= 22\,\mathfrak{o}_S \times \mathfrak{o}_S.\end{aligned}$$

We then observe that the action of $e^{\pi^2_S} := S \times S + \pi^2_S + \frac{1}{2}\pi^2_S \cdot \pi^2_S = S \times S + \pi^2_S + 11\,\mathfrak{o}_S \times \mathfrak{o}_S$ on the Chow group $\mathrm{CH}^*(S)$, called the *Fourier transform* and denoted $\mathcal{F}$, induces the same splitting as the Chow–Künneth decomposition considered above. Indeed, writing

$$\mathrm{CH}^i(S)_s := \{\sigma \in \mathrm{CH}^i(S) : \mathcal{F}(\sigma) \in \mathrm{CH}^{2-i+s}(S)\},$$

we also have

$$\mathrm{CH}^i(S)_s = (\pi^{2i-s}_S)_*\mathrm{CH}^i(S).$$

With these notations, we then have $\mathrm{CH}^2(S) = \mathrm{CH}^2(S)_0 \oplus \mathrm{CH}^2(S)_2$, $\mathrm{CH}^1(S) = \mathrm{CH}^1(S)_0$ and $\mathrm{CH}^0(S) = \mathrm{CH}^0(S)_0$, with the multiplicative property that

$$\mathrm{CH}^2(S)_0 = \mathrm{CH}^1(S)_0 \cdot \mathrm{CH}^1(S)_0 = \langle \mathfrak{o}_S \rangle.$$

The Beauville–Bogomolov form q_S on a K3 surface S is simply given by the cup-product on $\mathrm{H}^2(S,\mathbb{Q})$; its inverse q_S^{-1} defines an element of $\mathrm{H}^2(S,\mathbb{Q})\otimes\mathrm{H}^2(S,\mathbb{Q})$. By the Künneth formula, we may view q_S^{-1} as an element of $\mathrm{H}^4(S\times S,\mathbb{Q})$ that we denote $\mathfrak{B}$ and call the Beauville–Bogomolov class. The description above immediately gives that the cohomology class of π_S^2 in $\mathrm{H}^4(S\times S,\mathbb{Q})$ is $\mathfrak{B}$. Let us denote $\mathfrak{b} := \iota_\Delta^*\mathfrak{B}$ and $\mathfrak{b}_i = p_i^*\mathfrak{b}$, where $p_i : S\times S\to S$ is the i^{th} projection, $i=1,2$. On the one hand, a cohomological computation yields

$$\mathfrak{B} = [\Delta_S] - \frac{1}{22}(\mathfrak{b}_1+\mathfrak{b}_2).$$

On the other hand, the cycle $L := \Delta_S - \mathfrak{o}_S\times S - S\times\mathfrak{o}_S$, which was previously denoted π_S^2, satisfies $\iota_\Delta^* L = c_2(S) - 2\mathfrak{o}_S = 22\mathfrak{o}_S$. Thus not only does L lift $\mathfrak{B}$ to rational equivalence, but L also lifts the above equation satisfied by $\mathfrak{B}$ in the sense that

$$L = \Delta_S - \frac{1}{22}(l_1+l_2)$$

in the Chow group $\mathrm{CH}^2(S\times S)$, where $l := \iota_\Delta^* L$ and $l_i := p_i^* l$, $i=1,2$. Moreover, a cycle L that satisfies the equation $L = \Delta_S - \frac{1}{22}\left((p_1^*\iota_\Delta^*L) + (p_2^*\iota_\Delta^*L)\right)$ is unique. Indeed, if $L+\varepsilon$ is another cycle such that $L+\varepsilon = \Delta_S - \frac{1}{22}\left(p_1^*\iota_\Delta^*(L+\varepsilon)+p_2^*\iota_\Delta^*(L+\varepsilon)\right)$, then $\varepsilon = -\frac{1}{22}(p_1^*\iota_\Delta^*\varepsilon + p_2^*\iota_\Delta^*\varepsilon)$. Consequently, $\iota_\Delta^*\varepsilon = -\frac{1}{11}\iota_\Delta^*\varepsilon$. Thus $\iota_\Delta^*\varepsilon = 0$ and hence $\varepsilon = 0$.

We may then state a refinement of the main result of [**11**].

THEOREM (Beauville–Voisin, revisited). *Let S be a complex projective K3 surface. Then there exists a unique 2-cycle $L\in\mathrm{CH}^2(S\times S)$ such that $L = \Delta_S - \frac{1}{22}(l_1+l_2)$. Moreover, L represents the Beauville–Bogomolov class $\mathfrak{B}$ on S and the Fourier transform associated to L induces a splitting of the Chow ring* $\mathrm{CH}^*(S)$. □

C. HyperKähler varieties of K3[2]-type

HyperKähler manifolds are simply connected compact Kähler manifolds F such that $\mathrm{H}^0(F,\Omega_F^2)$ is spanned by a nowhere degenerate 2-form. This class of manifolds constitutes a natural generalization of the notion of a K3 surface in higher dimensions. Such manifolds are endowed with a canonical symmetric bilinear form $q_F : \mathrm{H}^2(F,\mathbb{Z})\otimes\mathrm{H}^2(F,\mathbb{Z})\to\mathbb{Z}$ called the Beauville–Bogomolov form. Passing to rational coefficients, its inverse q_F^{-1} defines an element of $\mathrm{H}^2(F,\mathbb{Q})\otimes\mathrm{H}^2(F,\mathbb{Q})$ and we define the *Beauville–Bogomolov class* $\mathfrak{B}$ in $\mathrm{H}^4(F\times F,\mathbb{Q})$ to be the class corresponding to q_F^{-1} via the Künneth formula. In this manuscript, we will mostly be concerned with projective hyperKähler manifolds and these will simply be called *hyperKähler varieties*. A hyperKähler variety is said to be of $\mathrm{K3}^{[n]}$*-type* if it is deformation equivalent to the Hilbert scheme of length-n subschemes on a K3 surface. A first result is that if F is a hyperKähler variety of $\mathrm{K3}^{[n]}$-type, then there is a cycle $L\in\mathrm{CH}^2(F\times F)$ defined in (53) whose cohomology class is $\mathfrak{B}\in\mathrm{H}^4(F\times F,\mathbb{Q})$; see Theorem 9.15.

C.1. The Fourier decomposition for the Chow groups. Given a hyperKähler variety F of $\mathrm{K3}^{[2]}$-type endowed with a cycle L with cohomology class $\mathfrak{B}$, we define a descending filtration $\mathrm{F}^\bullet$ on the Chow groups $\mathrm{CH}^i(F)$ as

(4)
$$\mathrm{F}^{2k+1}\mathrm{CH}^i(F) = \mathrm{F}^{2k+2}\mathrm{CH}^i(F) := \ker\{(L^{4-i+k})_* : \mathrm{F}^{2k}\mathrm{CH}^i(F)\to\mathrm{CH}^{4-i+2k}(F)\}.$$

The cohomological description of the powers of $[L] = \mathfrak{B}$ given in Proposition 1.3 shows that this filtration should be of Bloch–Beilinson type. We then define the *Fourier transform* as

$$\mathcal{F}(\sigma) = (p_2)_*(e^L \cdot p_1^*\sigma), \quad \text{for all } \sigma \in \mathrm{CH}^*(F).$$

In this work, we ask if there is a canonical choice of a codimension-2 cycle L on $F \times F$ lifting the Beauville–Bogomolov class $\mathfrak{B}$ such that the Fourier transform associated to L induces a splitting of the conjectural Bloch–Beilinson filtration on the Chow group $\mathrm{CH}^*(F)$ compatible with its ring structure given by intersection product. We give positive answers in the case when F is either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a cubic fourfold.

Consider now a projective hyperKähler manifold F of $\mathrm{K3}^{[2]}$-type and let $\mathfrak{B}$ denote its Beauville–Bogomolov class, that is, the inverse of its Beauville–Bogomolov form seen as an element of $\mathrm{H}^4(F \times F, \mathbb{Q})$. We define $\mathfrak{b} := \iota_\Delta^*\mathfrak{B}$ and $\mathfrak{b}_i := p_i^*\mathfrak{b}$. Proposition 1.3 shows that $\mathfrak{B}$ is uniquely determined up to sign by the following quadratic equation :

$$(5) \quad \mathfrak{B}^2 = 2\,[\Delta_F] - \frac{2}{25}(\mathfrak{b}_1 + \mathfrak{b}_2)\cdot\mathfrak{B} - \frac{1}{23\cdot 25}(2\mathfrak{b}_1^2 - 23\mathfrak{b}_1\mathfrak{b}_2 + 2\mathfrak{b}_2^2) \quad \text{in } \mathrm{H}^8(F\times F, \mathbb{Q}).$$

We address the following ; see also Conjecture 2.1 for a more general version involving hyperKähler fourfolds whose cohomology ring is generated by degree-2 classes.

CONJECTURE 1. *Let F be a hyperKähler variety F of $\mathrm{K3}^{[2]}$-type. Then there exists a cycle $L \in \mathrm{CH}^2(F \times F)$ with cohomology class $\mathfrak{B} \in \mathrm{H}^4(F \times F, \mathbb{Q})$ satisfying*

$$(6) \quad L^2 = 2\,\Delta_F - \frac{2}{25}(l_1 + l_2)\cdot L - \frac{1}{23\cdot 25}(2l_1^2 - 23l_1l_2 + 2l_2^2) \quad \textit{in } \mathrm{CH}^4(F\times F),$$

where by definition we have set $l := \iota_\Delta^* L$ and $l_i := p_i^* l$.

In fact, we expect the symmetric cycle L defined in (53) to satisfy the quadratic equation (6). Moreover, we expect a symmetric cycle $L \in \mathrm{CH}^2(F)$ representing the Beauville–Bogomolov class $\mathfrak{B}$ to be uniquely determined by the quadratic equation (6) ; see Proposition 3.4 for some evidence. From now on, when F is the Hilbert scheme of length-2 subschemes on a K3 surface, F is endowed with the cycle L defined in (92) – it agrees with the one defined in (53) by Proposition 16.1. When F is the variety of lines on a cubic fourfold, F is endowed with the cycle L defined in (107) – although we do not give a proof, this cycle (107) should agree with the one defined in (53).

Our first result, upon which our work is built, is the following theorem ; see Theorem 14.5 and Theorem 19.2.

THEOREM 1. *Let F be either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a cubic fourfold. Then Conjecture 1 holds for F.*

Let us introduce the cycle $l := \iota_\Delta^* L \in \mathrm{CH}^2(F)$ – it turns out that $l = \frac{5}{6}c_2(F)$ when F is either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a cubic fourfold ; see (93) and (108). The following hypotheses, together with (6), constitute the key relations towards establishing a Fourier decomposition for the Chow groups of F :

$$L_*l^2 = 0 \; ; \tag{7}$$

$$L_*(l \cdot L_*\sigma) = 25\, L_*\sigma \qquad \text{for all } \sigma \in \mathrm{CH}^4(F) \; ; \tag{8}$$

$$(L^2)_*(l \cdot (L^2)_*\tau) = 0 \qquad \text{for all } \tau \in \mathrm{CH}^2(F). \tag{9}$$

Indeed, Theorem 2 below shows that in order to establish the existence of a Fourier decomposition on the Chow groups of a hyperKähler variety of K3$^{[2]}$-type, it suffices to show that the cycle L defined in (53) (which is a characteristic class of Markman's twisted sheaf [**35**]) satisfies (6), (7), (8) and (9). Note that Properties (8) and (9) describe the intersection of l with 2-cycles on F.

THEOREM 2. *Let F be a hyperKähler variety of K3$^{[2]}$-type. Assume that there exists a cycle $L \in \mathrm{CH}^2(F \times F)$ representing the Beauville–Bogomolov class $\mathfrak{B}$ satisfying* (6), (7), (8) *and* (9). *For instance, F could be either the Hilbert scheme of length-2 subschemes on a K3 surface endowed with the cycle L of* (92) *or the variety of lines on a cubic fourfold endowed with the cycle L of* (107). *Denote*

$$\mathrm{CH}^i(F)_s := \{\sigma \in \mathrm{CH}^i(F) : \mathcal{F}(\sigma) \in \mathrm{CH}^{4-i+s}(F)\}.$$

Then the Chow groups of F split canonically as

$$\begin{aligned}
\mathrm{CH}^0(F) &= \mathrm{CH}^0(F)_0 \; ; \\
\mathrm{CH}^1(F) &= \mathrm{CH}^1(F)_0 \; ; \\
\mathrm{CH}^2(F) &= \mathrm{CH}^2(F)_0 \oplus \mathrm{CH}^2(F)_2 \; ; \\
\mathrm{CH}^3(F) &= \mathrm{CH}^3(F)_0 \oplus \mathrm{CH}^3(F)_2 \; ; \\
\mathrm{CH}^4(F) &= \mathrm{CH}^4(F)_0 \oplus \mathrm{CH}^4(F)_2 \oplus \mathrm{CH}^4(F)_4.
\end{aligned}$$

Moreover, we have

$$\mathrm{CH}^i(F)_s = \mathrm{Gr}^s_{\mathrm{F}^\bullet}\mathrm{CH}^i(F), \text{ where } \mathrm{F}^\bullet \text{ denotes the filtration (4) ;}$$

and

$$\sigma \text{ belongs to } \mathrm{CH}^i(F)_s \text{ if and only if } \mathcal{F}(\sigma) \text{ belongs to } \mathrm{CH}^{4-i+s}(F)_s.$$

We give further evidence that the splitting obtained in Theorem 2 is the splitting of a conjectural filtration $\mathrm{F}^\bullet$ on $\mathrm{CH}^*(F)$ of Bloch–Beilinson type by showing that it arises as the splitting of a filtration induced by a Chow–Künneth decomposition of the diagonal ; see Theorem 3.3. Note that our indexing convention is such that the graded piece of the conjectural Bloch–Beilinson filtration $\mathrm{CH}^i(F)_s = \mathrm{Gr}^s_{\mathrm{F}}\mathrm{CH}^i(F)$ should only depend on the cohomology group $\mathrm{H}^{2i-s}(F, \mathbb{Q})$, or rather, on the Grothendieck motive $\mathfrak{h}^{2i-s}_{\mathrm{hom}}(F)$.

The proof that the Chow groups of a hyperKähler variety of K3$^{[2]}$-type satisfying hypotheses (6), (7), (8) and (9) have a Fourier decomposition as described in the conclusion of Theorem 2 is contained in Theorems 2.2 & 2.4. That the Hilbert scheme of length-2 subschemes on a K3 surface endowed with the cycle L of (92) satisfies hypotheses (6), (7), (8) and (9) is Theorem 14.5, Proposition 14.6 and Proposition 14.8. That the variety of lines on a cubic fourfold endowed with the cycle L of (107) satisfies hypotheses (6), (7), (8) and (9) is Theorem 19.2, Proposition 19.4 and Proposition 19.6.

Theorem 2 is of course reminiscent of the case of abelian varieties where the Fourier decomposition on the Chow groups is induced by the exponential of the Poincaré bundle. Beauville's proof relies essentially on the interplay of the Poincaré line bundle and the multiplication-by-n homomorphisms. Those homomorphisms are used in a crucial way to prove the compatibility of the Fourier decomposition with the intersection product. The difficulty in the case of hyperKähler varieties is that there are no obvious analogues to the multiplication-by-n morphisms ; see however Remark 21.11. Still, when F is the variety of lines on a cubic fourfold, Voisin [**55**] defined a rational self-map $\varphi : F \dashrightarrow F$ as follows. For a general point $[l] \in F$ representing a line l on X, there is a unique plane Π containing l which is tangent to X along l. Thus $\Pi \cdot X = 2l + l'$, where l' is the residue line. Then one defines $\varphi([l]) = [l']$. In Chapter 21, we study the graph of φ in depth and completely determine its class both modulo rational equivalence and homological equivalence. It turns out that the action of φ on the Chow groups of F respects the Fourier decomposition ; see Section 21.6. Thus in many respects the rational map φ may be considered as an "endomorphism" of F. The interplay of φ with L is used to prove many features of the Fourier decomposition on the Chow groups of F.

C.2. The Fourier decomposition for the Chow ring. As in the case of abelian varieties or K3 surfaces, we are interested in the compatibility of the Fourier decomposition of Theorem 2 with the ring structure of $\mathrm{CH}^*(F)$. In the hyperKähler case, this was initiated by Beauville [**9**] who considered the sub-ring of $\mathrm{CH}^*(F)$ generated by divisors on the Hilbert scheme of length-2 subschemes on a K3 surface, and then generalized by Voisin [**56**] who considered the sub-ring of $\mathrm{CH}^*(F)$ generated by divisors and the Chern classes of the tangent bundle when F is either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a cubic fourfold :

THEOREM (Beauville [**9**], Voisin [**56**]). *Let F be either the Hilbert scheme of length-2 subschemes on a K3 surface, or the variety of lines on a cubic fourfold. Then any polynomial expression $P(D_i, c_2(F))$, $D_i \in \mathrm{CH}^1(F)$, which is homologically trivial vanishes in the Chow ring* $\mathrm{CH}^*(F)$.

This theorem implies the existence of a zero-cycle $\mathfrak{o}_F \in \mathrm{CH}_0(F)$ which is the class of a point such that

$$\langle \mathfrak{o}_F \rangle = \langle c_2(F)^2 \rangle = \langle c_2(F) \rangle \cdot \mathrm{CH}^1(F)^{\cdot 2} = \mathrm{CH}^1(F)^{\cdot 4}.$$

This latter result can already be restated, in the context of our Fourier decomposition, as follows ; see Theorem 4.6.

$$\mathrm{CH}^4(F)_0 = \langle l^2 \rangle = \langle l \rangle \cdot \mathrm{CH}^1(F)_0^{\cdot 2} = \mathrm{CH}^1(F)_0^{\cdot 4}.$$

We ask whether

$$\mathrm{CH}^i(F)_s \cdot \mathrm{CH}^j(F)_r \subseteq \mathrm{CH}^{i+j}(F)_{r+s}, \quad \text{for all } (i,s), (j,r).$$

The following theorem answers this question affirmatively when F is the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a very general cubic fourfold. Hilbert schemes of length-2 subschemes on K3 surfaces are dense in the moduli of hyperKähler varieties of $\mathrm{K3}^{[2]}$-type, and the varieties of lines on cubic fourfolds form an algebraic component of maximal dimension. Therefore Theorem 3 gives strong evidence that a Fourier decomposition on the Chow ring of hyperKähler varieties of $\mathrm{K3}^{[2]}$-type should exist.

THEOREM 3. *Let F be either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a very general cubic fourfold. Then*

$$\mathrm{CH}^i(F)_s \cdot \mathrm{CH}^j(F)_r \subseteq \mathrm{CH}^{i+j}(F)_{r+s}, \quad \textit{for all } (i,s),(j,r).$$

Moreover equality holds except when $\mathrm{CH}^{i+j}(F)_{r+s} = \mathrm{CH}^3(F)_2$ *or* $\mathrm{CH}^2(F)_0$.

We actually show Theorem 3 in a few more cases. Indeed, it is shown in Theorem 6.5 that the existence of a Fourier decomposition on the Chow groups or Chow ring of a hyperKähler variety of K3$^{[2]}$-type is a birational invariant, so that the conclusion of Theorem 3 also holds for any hyperKähler fourfold that is birational to the Hilbert scheme of length-2 subschemes on a K3 surface or to the variety of lines on a very general cubic fourfold.

The proof of Theorem 3 uses in an essential way a Theorem of Beauville–Voisin [**11**] on the vanishing of the "modified diagonal" of the K3 surface S in the case when F is the Hilbert scheme $S^{[2]}$, while it uses in an essential way the rational self-map $\varphi : F \dashrightarrow F$ constructed by Voisin [**55**] when F is the variety of lines on a cubic fourfold. Theorem 3 is proved for divisors in Chapter 4, while it is proved in full generality for $S^{[2]}$ in Theorem 15.8 and for the variety of lines on a very general cubic fourfold in Chapter 22. In the $S^{[2]}$ case, we prove in fact a stronger result : the Fourier decomposition is induced by a *multiplicative Chow–Künneth decomposition* of the diagonal in the sense of Definition 8.1 ; see Theorem 15.8.

In addition to stating the multiplicativity property of the Fourier decomposition, Theorem 3 also states that $\mathrm{CH}^2(F)_2 \cdot \mathrm{CH}^2(F)_2 = \mathrm{CH}^4(F)_4$. This equality, which also holds for the variety of lines on a (not necessarily very general) cubic fourfold, reflects at the level of Chow groups, as predicted by the Bloch–Beilinson conjectures, the fact that the transcendental part of the Hodge structure $\mathrm{H}^4(F,\mathbb{Q})$ is a sub-quotient of $\mathrm{Sym}^2\,\mathrm{H}^2(F,\mathbb{Q})$. A proof in the case of $S^{[2]}$ can be found in Proposition 12.9 and a proof in the case of the variety of lines on a cubic fourfold can be found in Proposition 20.3. It is also expected from the Bloch–Beilinson conjectures that $\mathrm{CH}^2(F)_{0,\mathrm{hom}} = 0$; see Theorem 3.3. In fact, for $F = S^{[2]}$, this would essentially follow from the validity of Bloch's conjecture for S. Although we cannot prove such a vanishing, a direct consequence of Theorem 3 is that

$$\mathrm{CH}^1(F) \cdot \mathrm{CH}^2(F)_{0,\mathrm{hom}} = 0 \quad \text{and} \quad \mathrm{CH}^2(F)_0 \cdot \mathrm{CH}^2(F)_{0,\mathrm{hom}} = 0.$$

Propositions 22.4 & 22.5 show that the above identities also hold when F is the variety of lines on a (not necessarily very general) cubic fourfold.

Furthermore, we ought to mention that each piece of the decomposition of $\mathrm{CH}^i(F)$ obtained in Theorem 2 is non-trivial. Indeed, we have $\mathrm{CH}^4(F)_0 = \langle l^2 \rangle$ and $l\cdot : \mathrm{CH}^2(F)_2 \to \mathrm{CH}^4(F)$ is injective with image $\mathrm{CH}^4(F)_2$; see Theorems 2.2 & 2.4. Therefore, $\mathrm{CH}^4(F)_0 \oplus \mathrm{CH}^4(F)_2$ is supported on a surface. Since $\mathrm{H}^4(F, \mathcal{O}_F) \neq 0$, it follows from Bloch–Srinivas [**14**] that $\mathrm{CH}^4(F)_4 \neq 0$. Thus, because $\mathrm{CH}^2(F)_2 \cdot \mathrm{CH}^2(F)_2 = \mathrm{CH}^4(F)_4$, $\mathrm{CH}^2(F)_2 \neq 0$ and hence $\mathrm{CH}^4(F)_2 \neq 0$.

A direct consequence of Theorems 2 & 3 for zero-dimensional cycles is the following theorem which is analogous to the decomposition of the Chow group of zero-cycles on an abelian variety as can be found in [**8**, Proposition 4].

THEOREM 4. *Let F be either the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a cubic fourfold. Then*

$$\mathrm{CH}^4(F) = \langle l^2 \rangle \oplus \langle l \rangle \cdot L_*\mathrm{CH}^4(F) \oplus (L_*\mathrm{CH}^4(F))^{\cdot 2}.$$

Moreover, this decomposition agrees with the Fourier decomposition of Theorem 2.

Let us point out that, with the notations of Theorem 2, $L_*\mathrm{CH}^4(F) = \mathrm{CH}^2(F)_2$. A hyperKähler variety is simply connected. Thus its first Betti number vanishes and a theorem of Rojtman [**45**] implies that $\mathrm{CH}^4_{\mathbb{Z}}(F)_{\mathrm{hom}}$ is uniquely divisible. Therefore, Theorem 4 can actually be stated for 0-dimensional cycles with integral coefficients :

$$\mathrm{CH}^4_{\mathbb{Z}}(F) = \mathbb{Z}\mathfrak{o}_F \oplus \langle l \rangle \cdot (L_*\mathrm{CH}^4(F)) \oplus (L_*\mathrm{CH}^4(F))^{\cdot 2}.$$

Finally, let us mention the following remark which could be useful to future work. As it is not essential to the work presented here, the details are not expounded. Given a divisor D on F with $q_F([D]) \neq 0$ where q_F denotes the quadratic form attached to the Beauville–Bogomolov form, the cycle $L_D := L - \frac{1}{q_F([D])} D_1 \cdot D_2$, where $D_i := p_i^*D$, $i = 1, 2$, defines a *special Fourier transform* $\mathcal{F}_D : \mathrm{CH}^*(F) \to \mathrm{CH}^*(F), x \mapsto (p_2)_*(e^{L_D} \cdot p_1^*x)$ such that $\mathcal{F}_D \circ \mathcal{F}_D$ induces a further splitting of $\mathrm{CH}^*(F)$ which takes into account D. For instance, if F is the variety of lines on a cubic fourfold and if g is the Plücker polarization on F, then $\mathcal{F}_g \circ \mathcal{F}_g$ induces an orthogonal decomposition $\mathrm{CH}^1(F) = \langle g \rangle \oplus \mathrm{CH}^1(F)_{\mathrm{prim}}$ but also a further decomposition $\mathrm{CH}^3(F)_2 = A \oplus B$, where $\mathcal{F}_g \circ \mathcal{F}_g$ acts as the identity on A and as zero on $B = g \cdot \mathrm{CH}^2(F)_2$. It turns out that φ^* acts by multiplication by 4 on A and by multiplication by -14 on B.

D. HyperKähler varieties

Let F be a hyperKähler variety. In general, the sub-algebra of $\mathrm{H}^*(F \times F, \mathbb{Q})$ generated by the Beauville–Bogomolov class $\mathfrak{B}$, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ only "sees" the sub-Hodge structure of $\mathrm{H}^*(F, \mathbb{Q})$ generated by $\mathrm{H}^2(F, \mathbb{Q})$. Precisely, it can be checked that the cohomological Fourier transform $\mathcal{F}$ acts trivially on the orthogonal complement of the image of $\mathrm{Sym}^n \mathrm{H}^2(F, \mathbb{Q})$ inside $\mathrm{H}^{2n}(F, \mathbb{Q})$. Therefore, it does not seem possible to formulate directly an analogous Fourier decomposition, as that of Theorems 2 & 3, for the Chow ring of those hyperKähler varieties F whose cohomology is not generated by $\mathrm{H}^2(F, \mathbb{Q})$. There are however three questions that we would like to raise concerning algebraic cycles on hyperKähler varieties.

D.1. Zero-cycles on hyperKähler varieties. If ω is a nowhere degenerate 2-form on a hyperKähler variety F, then the powers $\omega^n \in \mathrm{Sym}^n \mathrm{H}^2(F, \mathbb{C})$ of ω span the degree-zero graded part of the cohomology of F for the coniveau filtration. Hence we can expect that if a canonical 2-cycle $L \in \mathrm{CH}^2(F \times F)$ representing the Beauville–Bogomolov class $\mathfrak{B}$ exists, then the Fourier transform $\mathcal{F} := e^L$ splits a Bloch–Beilinson type filtration on $\mathrm{CH}_0(F)$. It is thus tempting to ask whether the decomposition of Theorem 4 holds for hyperKähler varieties :

CONJECTURE 2. *Let F be a hyperKähler variety of dimension $2d$. Then there exists a canonical cycle $L \in \mathrm{CH}^2(F \times F)$ representing the Beauville–Bogomolov* *class* $\mathfrak{B}$ *which induces a canonical splitting*

$$\mathrm{CH}^{2d}(F) = \bigoplus_{s=0}^{d} \mathrm{CH}^{2d}(F)_{2s},$$

where $\mathrm{CH}^{2d}(F)_{2s} := \{\sigma \in \mathrm{CH}^{2d}(F) : \mathcal{F}(\sigma) \in \mathrm{CH}^{2s}(F)\}$. *Moreover, we have*

$$\mathrm{CH}^{2d}(F)_{2s} = \langle l^{d-s}\rangle \cdot (L_*\mathrm{CH}^{2d}(F))^{\cdot s}$$

and

$$\mathrm{CH}^{2d}(F)_{2s} \supseteq P(l, D_1, \ldots, D_r) \cdot (L_*\mathrm{CH}^{2d}(F))^{\cdot s},$$

for any degree $2d-2s$ *weighted homogeneous polynomial* P *in* l *and divisors* D_i, $i = 1, \ldots, r$.

Note that in the case of hyperKähler varieties of $\mathrm{K3}^{[n]}$-type, a candidate for a canonical cycle L representing $\mathfrak{B}$ is given by Theorem 9.15. Note also that, in general, because $[L] = \mathfrak{B}$ induces an isomorphism $\mathrm{H}^{2d-2}(F,\mathbb{Q}) \to \mathrm{H}^2(F,\mathbb{Q})$ a consequence of the Bloch–Beilinson conjectures would be that $L_*\mathrm{CH}^{2d}(F) = \ker\{AJ^2 : \mathrm{CH}^2(F)_{\mathrm{hom}} \to J^2(F)\otimes\mathbb{Q}\}$, where AJ^2 denotes Griffiths' Abel–Jacobi map tensored with $\mathbb{Q}$.

D.2. On the existence of a distinguished cycle $L \in \mathrm{CH}^2(F\times F)$. Let F be a hyperKähler variety. Beauville [**9**] conjectured that the sub-algebra V_F of $\mathrm{CH}^*(F)$ generated by divisors injects into cohomology via the cycle class map. Voisin [**56**] conjectured that if one adds to V_F the Chern classes of the tangent bundle of F, then the resulting sub-algebra still injects into cohomology. The following conjecture, which is in the same vein as the conjecture of Beauville–Voisin, is rather speculative but as Theorem 5 shows, it implies the existence of a Fourier decomposition on the Chow ring of hyperKähler fourfolds of $\mathrm{K3}^{[2]}$-type. Before we can state it, we introduce some notations. Let X be a smooth projective variety, and let the $\mathbb{Q}$-vector space $\bigoplus_n \mathrm{CH}^*(X^n)$ be equipped with the algebra structure given by intersection product on each summand. Denote $p_{X^n\,;i_1,\ldots,i_k} : X^n \to X^k$ the projection on the $(i_1,\ldots,i_k)^{\mathrm{th}}$ factor for $1 \le i_1 < \ldots < i_k \le n$, and $\iota_{\Delta,X^n} : X \to X^n$ the diagonal embedding. Given cycles $\sigma_1,\ldots,\sigma_r \in \bigoplus_n \mathrm{CH}^*(X^n)$, we define $V(X\,;\sigma_1,\ldots,\sigma_r)$ to be the smallest sub-algebra of $\bigoplus_n \mathrm{CH}^*(X^n)$ that contains $\sigma_1,\ldots,\sigma_r$ and that is stable under $(p_{X^n\,;i_1,\ldots,i_k})_*$, $(p_{X^n\,;i_1,\ldots,i_k})^*$, $(\iota_{\Delta,X^n})_*$ and $(\iota_{\Delta,X^n})^*$.

CONJECTURE 3 (Generalization of Conjecture 1). *Let F be a hyperKähler variety. Then there exists a canonical cycle $L \in \mathrm{CH}^2(F\times F)$ representing the Beauville–Bogomolov class $\mathfrak{B} \in \mathrm{H}^4(F\times F,\mathbb{Q})$ such that the restriction of the cycle class map $\bigoplus_n \mathrm{CH}^*(F^n) \to \bigoplus_n \mathrm{H}^*(F^n,\mathbb{Q})$ to $V(F\,;L, c_0(F), c_2(F),\ldots,c_{2d}(F), D_1,\ldots,D_r)$, for any $D_i \in \mathrm{CH}^1(F)$, is injective.*

Note that in higher dimensions l and $c_2(F)$ are no longer proportional in $\mathrm{H}^4(F,\mathbb{Q})$. This is the reason why we consider $V(F\,;L,c_2(F),\ldots,c_{2d}(F),D_1,\ldots,D_r)$ rather than $V(F\,;L,D_1,\ldots,D_r)$. Conjecture 3 is very strong : Voisin [**56**, Conjecture 1.6] had already stated it in the case of K3 surfaces (in that case L need not be specified as it is simply given by $\Delta_S - \mathfrak{o}_S\times S - S\times\mathfrak{o}_S$) and noticed [**59**, p.92] that it implies the finite dimensionality in the sense of Kimura [**33**] and O'Sullivan [**42**] of the Chow motive of S. The following theorem reduces the Fourier decomposition problem for the Chow ring of hyperKähler varieties of $K3^{[2]}$-type to a weaker form of Conjecture 3 that only involves L.

THEOREM 5 (Theorem 8.18). *Let F be a hyperKähler variety of $K3^{[2]}$-type. Assume that F satisfies the following weaker version of Conjecture 3 : there exists a cycle $L \in \mathrm{CH}^2(F \times F)$ representing the Beauville–Bogomolov class $\mathfrak{B}$ satisfying equation (6), and the restriction of the cycle class map $\bigoplus_n \mathrm{CH}^*(F^n) \to \bigoplus_n \mathrm{H}^*(F^n, \mathbb{Q})$ to $V(F\,;L)$ is injective. Then $\mathrm{CH}^*(F)$ admits a Fourier decomposition as in Theorem 2 which is compatible with its ring structure.*

This provides an approach to proving the Fourier decomposition for the Chow ring of hyperKähler varieties of $K3^{[2]}$-type that would avoid having to deal with non-generic cycles. Conjecture 3 can be considered as an analogue for hyperKähler varieties of O'Sullivan's theorem [**43**] which is concerned with abelian varieties. In the same way that Conjecture 3 implies the existence of a Fourier decomposition for the Chow ring of hyperKähler varieties of $K3^{[2]}$-type, we explain in Section 7 how Beauville's Fourier decomposition theorem for abelian varieties can be deduced directly from O'Sullivan's theorem.

Finally, as already pointed out, the diagonal $[\Delta_F]$ cannot be expressed as a polynomial in $\mathfrak{B}$, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ when F is a hyperKähler variety whose cohomology ring is not generated by degree-2 cohomology classes. Nevertheless, the orthogonal projector on the sub-Hodge structure generated by $\mathrm{H}^2(F, \mathbb{Q})$ can be expressed as a polynomial in $\mathfrak{B}$, $\mathfrak{b}_1$ and $\mathfrak{b}_2$. If one believes in Conjecture 3, this projector should in fact lift to a projector, denoted Π, modulo rational equivalence. In that case, it seems reasonable to expect $\Pi_*\mathrm{CH}^*(F)$ to be a sub-ring of $\mathrm{CH}^*(F)$ and to expect the existence of a Fourier decomposition with kernel L on the ring $\Pi_*\mathrm{CH}^*(F)$.

D.3. Multiplicative Chow–Künneth decompositions. We already mentioned that the Fourier decomposition for the Chow ring of the Hilbert scheme of length-2 subschemes on a K3 surface or the variety of lines on a very general cubic fourfold is in fact induced by a Chow–Künneth decomposition of the diagonal (*cf.* Chapter 3 for a definition). A smooth projective variety X of dimension d is said to admit a *weakly multiplicative Chow–Künneth decomposition* if it can be endowed with a Chow–Künneth decomposition $\{\pi_X^i : 0 \le i \le 2d\}$ that induces a decomposition of the Chow ring of X. That is, writing

$$\mathrm{CH}^i_{\mathrm{CK}}(X)_s := (\pi_X^{2i-s})_*\mathrm{CH}^i(X),$$

we have

$$\mathrm{CH}^i_{\mathrm{CK}}(X)_s \cdot \mathrm{CH}^j_{\mathrm{CK}}(X)_r \subseteq \mathrm{CH}^{i+j}_{\mathrm{CK}}(X)_{r+s}, \quad \text{for all } (i,s),\ (j,r).$$

The Chow–Künneth decomposition is said to be *multiplicative* if the above holds at the level of correspondences ; see Definition 8.1.

Together with Murre's conjecture (D) as stated in Chapter 3, we ask :

CONJECTURE 4. *Let X be a hyperKähler variety. Then X can be endowed with a Chow–Künneth decomposition that is multiplicative. Moreover, the cycle class map $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X, \mathbb{Q})$ restricted to $\mathrm{CH}^i_{\mathrm{CK}}(X)_0$ is injective.*

Note that if A is an abelian variety, then A has a multiplicative Chow–Künneth decomposition (*cf.* Example 8.3) ; and that the cycle class map restricted to the