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Analytic Methods for Partial Differential Equations

偏微分方程解析方法

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With 25 Figures

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To our past and present students without whom this work would not have been developed

Preface

The subject of partial differential equations holds an exciting and special position in mathematics. Partial differential equations were not consciously created as a subject but emerged in the 18th century as ordinary differential equations failed to describe the physical principles being studied. The subject was originally developed by the major names of mathematics, in particular, Leonard Euler and Joseph-Louis Lagrange who studied waves on strings; Daniel Bernoulli and Euler who considered potential theory, with later developments by Adrien-Marie Legendre and Pierre-Simon Laplace; and Joseph Fourier's famous work on series expansions for the heat equation. Many of the greatest advances in modern science have been based on discovering the underlying partial differential equation for the process in question. James Clerk Maxwell, for example, put electricity and magnetism into a unified theory by establishing Maxwell's equations for electromagnetic theory, which gave solutions for problems in radio wave propagation, the diffraction of light and X-ray developments. Schrödinger's equation for quantum mechanical processes at the atomic level leads to experimentally verifiable results which have changed the face of atomic physics and chemistry in the 20th century. In fluid mechanics, the Navier–Stokes' equations form a basis for huge number-crunching activities associated with such widely disparate topics as weather forcasting and the design of supersonic aircraft.

Inevitably the study of partial differential equations is a large undertaking, and falls into several areas of mathematics. At one extreme the main interest is in the existence and uniqueness of solutions, and the functional analysis of the proofs of these properties. At the other extreme, lies the applied mathematical and engineering quest to find useful solutions, either analytically or numerically, to these important equations which can be used in design and construction. In both this text, and the companion volume (Evans, 1999), the emphasis is on the practical solution rather than the theoretical background, though this important work is recognised by pointers to further reading. This approach is based on courses given by the authors while at De Montfort University.

Hence in the first chapter, we start by covering some of the mathematical background including orthogonal polynomials, special functions such as Legendre Polynomials and Bessel functions and a brief coverage of complex variables. The use of characteristics to classify partial differential equations leads to specific techiques in the following chapters. This is supported by brief derivations of the wave equation, the heat equation and Laplace's equation. The chapter is concluded with some background to generalised functions for use in the final chapter on Green's functions.

Chapter 2 is a conventional coverage of separation of variables, applied to the heat equation and Laplace's equation in Cartesian, cylindrical polar and spherical polar coordinates. Chapter 3 is concerned with solutions involving characteristic curves, and seemed the natural place for first-order equations, including Charpit's method for nonlinear first-order equations. The chapter then moves on to second-order equations and D'Alembert's solution of the wave equation, including the method of characteristics in an analytic setting.

Integral transforms are covered in Chapter 4, with work on Fourier's integral theorem, Fourier sine and cosine transforms, Fourier complex transforms and Laplace transforms.

The final chapter is on Green's functions, and perforce covers the basic work in this field only. We have of course Green's birth place (Sneinton Windmill) and his grave very near to us here. In all these chapters, space limitations had to be considered and some cuts were made to this end. Topics here include Green's functions for the wave equation, the diffusion equation and Laplace's equation; Helmholtz and Schrödinger's equations with applications to scattering theory; Maxwell's equations; and Green's functions in optics with Kirchhoff diffraction theory. Approximation methods and Born series are also considered briefly.

Most sections have a set of exercises, and fairly complete solutions can be found in the appendix. Exceptions are small introductory sections and where a second section is required to make a topic viable for solution and further investigation by the reader. The exercises and solutions form an important part of the book and provide much insight to the ideas introduced in the text.

In the last stages of the preparation, the completed manuscript was read by Endre Süli (Oxford University), and we are very grateful for his general remarks and detailed comments.

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Contents

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Mathematical Preliminaries

1.1 Introduction

Partial differential equations emerged when shortcomings with the use of ordinary differential equations were found in the study of vibrations of strings, propagation of sound, waves in liquids and in gravitational attraction. Originally the calculus of partial derivatives was supplied by Euler in a series of papers concerned with hydrodynamics in 1734. This work was extended by D'Alembert in 1744 and 1745 in connection with the study of dynamics.

Partial differential equations are the basis of almost every branch of applied mathematics. Such equations arise from mathematical models of most real life situations. Hence quantum mechanics depends on Schrödinger's equations, fluid mechanics on various forms of Navier–Stokes' equations and electromagnetic theory on Maxwell's equations. Partial differential equations form a very large area of study in mathematics, and are therefore important for both analytical and numerical considerations. The analytical aspects are covered in this text and the numerical aspects in the companion volume, "Numerical methods for partial differential equations".

Inevitably there are many aspects of other branches of mathematics which are pertinent to this work, and the relevant material has been brought together in this chapter to save long digressions later, and to give an element of completeness. The first two sections should be covered at the first reading and form a general introduction to the book as a whole. The later sections deal with a range of related topics that will be needed later, and may be tackled as required.

When the differential equations involve only one independent variable such as $y(t)$ in the equation for simple harmonic motion given by

$$\frac{d^2y}{dt^2} + k^2 y = 0 \tag{1.1.1}$$

this is then called an ordinary differential equation. Standard methods are available for the analytic solution of particular classes of such equations such as those with constant coefficients, and these methods are familiar in references such as Nagle and Saff (1993), or the classic, Piaggio (1950). However, it is very easy to write an equation whose closed form solution is not expressible in simple terms such as

$$\frac{d^2y}{dx^2} = xy. \tag{1.1.2}$$

For such a problem the ordinary differential equation itself defines the solution function and is used to derive its analytic properties by such devices as series solutions. Numerical methods come into their own to obtain values of the solution function and again there is a vast literature on this topic which includes Lambert (1990) and Evans (1996).

Partial differential equations follow a similar line, but now the dependent variable is a function of more than one independent variable, and hence the derivatives are all partial derivatives. In view of ordinary differential equations, some types lend themselves to analytic solution, and there is a separate literature on numerical solutions. These aspects form the contents of this book and its companion volume.

The *order* of a partial differential equation is the order of the highest derivative. First-order equations can often be reduced to the solution of ordinary differential equations, which will be seen later in the considerations of characteristics. Second-order equations tend to demonstrate the numerical methods applicable to partial differential equations in general. For the most part, consideration here is limited to linear problems – the nonlinear ones constituting current research problems. Linear problems have the dependent variable and its partial derivatives occurring only to the first degree, hence there are no products of the dependent variable and its derivatives. Hence the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.1.3}$$

is linear. It is called Laplace's equation, and it will be a major topic in this book, whereas

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1.4}$$

is the nonlinear Korteweg–de Vries equation. For solutions of this equation, the method of inverse scattering is employed which is outside the scope of this book, and may be pursued in Ablowitz and Clarkson (1991). A linear equation is said to be *homogeneous* if each term contains either the dependent variable or one of its derivatives, otherwise it is said to be non-homogeneous or inhomogeneous.

The fundamental property of a homogeneous linear problem is that if f_1 and f_2 are solutions then so is $f_1 + f_2$. To begin the discussion, three specific physical applications which prove typical of the problems to be solved are introduced. A classification of second-order equations is then covered, and each of the three physical problems falls into a different class in this categorisation.

The first of these physical problems is the heat or diffusion equation which can be derived by considering an arbitrary volume V. The heat crossing the boundary will equate to the change of heat within the solid, which results in the equation

$$\int_V \rho c \frac{\partial \theta}{\partial t} dV = \int_S k \text{ grad } \theta \cdot d\mathbf{S} + \int_V H(\mathbf{r}, \theta, t)\, dV \tag{1.1.5}$$

where $d\mathbf{S} = \mathbf{n} dS$ with $\mathbf{n}$ the unit outward normal to the surface S of V and dS is a surface element, θ is the temperature, k is the thermal conductivity, ρ the density and c the specific heat. H represents any heat generated in the volume by such action as radioactive decay, electrical heating or chemical action. A short-hand notation, common in continuum mechanics is used here where grad is defined by

$$\text{grad } u = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\} \tag{1.1.6}$$

which generates a vector from a scalar u. This is often written in the condensed form-grad $u = \nabla u$. A further short-hand notation that will be used where a condensed notation is acceptable is the use of subscripts to denote partial derivatives. Hence the above definition will become

$$\text{grad } u = \{u_x, u_y, u_z\}. \tag{1.1.7}$$

With this definition, the z dependence may be absent in the case of partial differential equations in two independent variables which will be the dominant case in this book. There are two other *vector operators* which are also used in this book, namely div and curl, defined by

$$\text{div}\, \mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \tag{1.1.8}$$

and

$$\text{curl}\, \mathbf{a} = \nabla \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} \tag{1.1.9}$$

where $\mathbf{a} = \{a_1, a_2, a_3\}$. Hence the operator div operates on a vector to generate a scalar, and the operator curl operates on a vector and generates a vector. With these definitions in place the derivation of the main equations may now be continued.

The integral over the surface S can be converted to a volume integral by the divergence theorem (Apostol, 1974) to give

$$\int_V \rho c \frac{\partial \theta}{\partial t} dV = \int_V \text{div } (k \text{ grad } \theta) dV + \int_V H(\mathbf{r}, \theta, t)\, dV. \tag{1.1.10}$$

However, this balance is valid for an arbitrary volume and therefore the integrands themselves must satisfy

$$\rho c \frac{\partial \theta}{\partial t} = \text{div } (k \text{ grad } \theta) + H(\mathbf{r}, \theta, t). \qquad (1.1.11)$$

In the special but common case in which $k =$ constant, the diffusion equation reduces to

$$\frac{\partial \theta}{\partial t} = K\nabla^2\theta + Q(\mathbf{r}, \theta, t) \qquad (1.1.12)$$

with

$$\left. \begin{aligned} K &= \frac{k}{\rho c} \\ Q &= \frac{H}{\rho c} \end{aligned} \right\} .$$

Typical boundary conditions for this equation would include the following.

(i) $\theta(\mathbf{r}, t)$ is a prescribed function of t for every point $\mathbf{r}$ on the boundary surface S.

(ii) The normal flux through the boundary $\dfrac{\partial \theta}{\partial n}$ is prescribed on S where $\mathbf{n}$ is a normal vector to the surface S.

(iii) The surface radiation is defined over S, for example, by

$$\frac{\partial \theta}{\partial n} = -a(\theta - \theta_0) \qquad (1.1.13)$$

which is Newton's law of radiation.

The heat or diffusion equation applies to a very large number of other physical situations. The diffusion of matter such as smoke in the atmosphere, or a dye or pollutant in a liquid is governed by Fick's law

$$\mathbf{J} = -D \text{ grad } c \qquad (1.1.14)$$

where D is the coefficient of diffusion and c is the concentration. The vector $\mathbf{J}$ is the diffusion current vector, and therefore c satisfies

$$\frac{\partial c}{\partial t} = \text{div } (D \text{ grad } c) \qquad (1.1.15)$$

or

$$\frac{\partial c}{\partial t} = D\nabla^2 c \qquad (1.1.16)$$

if D is a constant. Other physical situations, which are modelled by the diffusion equation include neutron slowing, vorticity diffusion and propagation of long electromagnetic waves in a good conductor such as an aerial.

The second of the fundamental physical equations is the wave equation. Consider a small length of a stretched string as shown in Figure 1.1.

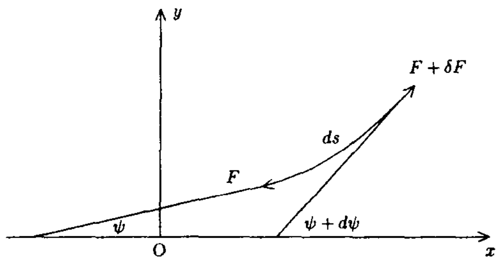

Fig. 1.1.

Then Newton's second law applied to the small length gives

$$F\sin(\psi + d\psi) - F\sin\psi = \rho\, ds\frac{\partial^2 y}{\partial t^2} = F\cos\psi d\psi \tag{1.1.17}$$

to first order where F is the tension in the string, ρ is the string density, ψ is the tangential angle of the string to the x-axis, s is the distance coordinate along the string and y is the displacement from the neutral position. From elementary calculus,

$$\tan\psi = \frac{\partial y}{\partial x} \tag{1.1.18}$$

and hence

$$\sec^2\psi d\psi = \frac{\partial^2 y}{\partial x^2}dx \tag{1.1.19}$$

which yields

$$\begin{aligned}\rho\frac{\partial^2 y}{\partial t^2} &= F\cos^3\psi\frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial s}\\ &= F\cos^4\psi\frac{\partial^2 y}{\partial x^2}\end{aligned} \tag{1.1.20}$$

where $\frac{\partial x}{\partial s} = \cos\psi$ as ψ is the angle the tangent makes with the x-axis. However, for oscillations of small amplitude

$$\cos^2\psi = \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{-1} \sim 1 \tag{1.1.21}$$

to yield the wave equation in the form

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{1.1.22}$$

with

$$c^2 = \frac{F}{\rho}. \tag{1.1.23}$$

The third important example is Laplace's equation which is based on Gauss's law in electromagnetism (see Atkin, 1962) namely

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \rho \tag{1.1.24}$$

where $\mathbf{E}$ is the electric field, and ρ is the charge density with S being the surface of an arbitary volume. If $\rho = 0$, then an application of the divergence theorem, gives the differential equation

$$\text{div } \mathbf{E} = 0. \tag{1.1.25}$$

However, Maxwell's equations with no time varying magnetic field yield

$$\text{curl } \mathbf{E} = 0 \tag{1.1.26}$$

which is the condition for the field to be irrotational. With this proviso, there exists a ϕ such that

$$\mathbf{E} = \text{ grad } \phi \tag{1.1.27}$$

and hence

$$\text{div grad } \phi = 0 \tag{1.1.28}$$

or

$$\nabla^2 \phi = 0 \tag{1.1.29}$$

which is Laplace's equation. The same equation holds for the flow of an ideal fluid. Such a fluid has no viscosity and being incompressible the equation of continuity is div $\mathbf{q} = 0$, where $\mathbf{q}$ is the flow velocity vector (see Acheson, 1990). For irrotational flows the equivalent of 1.1.26 holds to allow the use of the potential function $\mathbf{q} = \text{grad}\phi$ and again the defining equation is 1.1.29.

These three major physical problems (1.1.16, 1.1.21 and 1.1.28) are typical of the main types of second-order linear partial differential equations, and in the next section mathematical arguments will be used to establish a classification of such problems.

The following exercises cover the derivation of variations to the main equations above to allow further physical features to be considered. The mathematical and numerical solutions to these extended problems fall into the remit of the solutions for the basic equations.

EXERCISES

1.1 Establish that if a string is vibrating in the presence of air resistance which is proportional to the string velocity then the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - r\frac{\partial u}{\partial t} \qquad \text{with} \quad r > 0.$$

1.2 Show that if a vibrating string experiences a transverse elastic force (such as a vibrating coiled spring), then the relevant form of the wave equation is

$$\frac{\partial^2 t}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - ku \qquad \text{with} \quad k > 0.$$

1.3 If a vibrating string is subject to an external force which is defined by $f(x,t)$, then show that the wave equation takes the form

$$\frac{\partial^2 t}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t).$$

1.4 If there is an external source or sink of heat given by $f(x,t)$ (such as that due to electrical heating of a rod, chemical heating or radioactive heating), then the diffusion equation becomes

$$\frac{\partial u}{\partial t} = \nabla(k\nabla u) + f(x,t).$$

1.5 If the end of a vibrating string is in a viscous liquid then energy is radiated out from the string to the liquid. Show that the endpoint boundary condition has the form

$$\frac{\partial u}{\partial n} + b\frac{\partial u}{\partial t} = 0$$

where n is the normal derivative and b is a constant which is positive if energy is radiated to the liquid.

1.6 When a current flows along a cable with leakage, G, the loss of voltage is caused by resistance and inductance. The resistance loss is Ri where R is the resistance and i is the current (Ohm's Law). The inductance loss is proportional to the rate of change of current (Gauss's Law), which gives the term Li_t where L is the inductance. Hence, the voltage equation is

$$v_x + Ri + Li_t = 0.$$

The current change is due to capacitance C, and leakage G. These terms yield

$$i_x + Cv_t + Gv = 0.$$

Deduce the telegraph equation in the form

$$LC\frac{\partial^2 v}{\partial t^2} + (GL + RC)\frac{\partial v}{\partial t} + RGv = \frac{\partial^2 v}{\partial x^2}.$$

Find the equation satisfied by the current i.