Modern Methods in Partial Differential Equations

AN INTRODUCTION

MARTIN SCHECHTER

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PREFACE

The subject of partial differential operators is very broad and encompasses many aspects of analysis. We can convince ourselves of this fact merely by noting that the study of analytic functions of a complex variable is, in reality, the study of solutions of a simple system of partial differential equations (the Cauchy–Riemann equations). Moreover, in our present state of knowledge, we have only scratched the surface. The more involved scientific theory becomes, the more and varied the partial differential equations that arise. Other branches of mathematics also contribute their share. The types of partial differential equations and systems that are possible can stagger the imagination.

There seems to be a dichotomy in the types of books published on the topic of partial differential equations. One kind of presentation studies the equations of classical physics (the wave, heat, and potential energy equations) and employs mostly separation of variables and Fourier series. This part of the theory has been thoroughly studied and is near completion. There are many excellent books written on this topic. It can be referred to as the classical theory of partial differential equations, because the major part of it was known more than fifty years ago.

In the past thirty years there has been a surge of activity in new directions. Higher order equations and systems have been studied, including those that do not fall into previously defined categories. Even the concept of a solution has been broadened, and operators more general than partial differential operators have become popular.

The purpose of this book is to introduce the student to the modern techniques and methods that have been used in the newer theory. I want the student to get a taste and feeling for the powerful tools that are used, but I do not want him to be engrossed in the minute details that can hide the basic ideas behind the methods. I try to take the middle of the road—to attack problems of greater generality than those considered in the classical theory, but not to require the most general and refined machinery available. With respect to the latter, I try to introduce new tools sparingly, and to use only what is needed to obtain a meaningful (albeit not necessarily the most powerful) result. In this way I hope the student will be able to distinguish the trees from the forest.

Another reason for my keeping the background material to a minimum is to allow students to learn the subject at an earlier stage. The material in this book has been taught to first year graduate students at Yeshiva University over a ten year period. Moreover, I have endeavored to present it in such a way as to make it accessible to undergraduates as well. The reader should have a basic knowledge of advanced calculus. Lebesgue integration theory is not actually used. It is only noted that L^2 is complete. Anyone who is willing to accept this fact needs no more background in integration theory. The theory of analytic functions of a complex variable is used only in a few places, and even then only in an elementary way. All background material in Hilbert space and linear algebra is given where necessary. It is actually surprising that one can accomplish so much with so little.

A book on partial differential equations can hope to cover only a small fraction of the basic material that is known. The author must select, but there are no logical grounds upon which to base such a selection. Many authors choose what they feel are the most important topics, but the subjectivity of their choices is fairly obvious.

The present book is no different in this respect. I have chosen subject matter that I feel will motivate the student and introduce him to techniques that have wide applicability to many other problems, in partial differential equations as well as other branches of analysis. Also, I want to give the reader a fair idea of what is to be expected in other situations, and what methods can be used. In addition, I want a uniform theme and outlook; throughout the book I consider a single linear partial differential operator of arbitrary order. In each problem I look for existence, uniqueness, estimates, and regularity of solutions. I try to pick analytical tools that can be used throughout the book, and not only in one or two isolated instances. Whenever possible I try to use the same spaces of functions throughout (basically the L^2 spaces).

I have tried to give proper credit to various research articles used in obtaining material, but I am sure that I have benefited directly or indirectly from countless others. In addition I have added, to the bibliography, several books of interest on the modern theory of partial differential equations.

June, 1976 — Martin Schechter

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Martin Schechter

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LIST OF SYMBOLS

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CHAPTER

ONE

EXISTENCE OF SOLUTIONS

1-1 INTRODUCTION

A partial differential equation is, as the name implies, an equation containing a partial derivative. Of course, the derivative is to be taken of an unknown function of more than one variable (if the function were known, we could take the derivative and it would disappear; if it depended only on one variable, we would call the equation an ordinary differential equation). The simplest partial differential equation is

$$\frac{\partial u(x, y)}{\partial x} = 0 \tag{1-1}$$

where the unknown function u depends on two variables x, y. The solution of Eq. (1-1) is obviously

$$u(x, y) = g(y) \tag{1-2}$$

where $g(y)$ is any function of y alone. Although this example is fairly simple, we should examine it a bit more closely. First of all, what do we mean by a "solution" of Eq. (1-1)? You say, "That is obvious; we mean simply a function $u(x, y)$, which, when substituted into Eq. (1-1), makes the equation hold." However, a little reflection shows immediately that certain problems arise, albeit that for this particular equation they are easily solved.

We cannot just substitute any proposed solution into Eq. (1-1): we must start by differentiating it. Therefore, the first requirement we must impose on $u(x, y)$ in order that it be a solution of Eq. (1-1) is that it possess a derivative with respect to x. Second, for what values of x, y is Eq. (1-1) to hold? All real values, or just some? This certainly has to be specified. Next, let us examine our "solution," Eq. (1-2). What kind of function is $g(y)$? Must it possess a derivative with respect to y, or can it even be discontinuous? Or perhaps it need not be a function at all in the usual sense, but a so-called "distribution" (ignore this last statement if you have never heard the term).

Another observation is that no matter what kind of functions we admit, Eq. (1-1) will have many solutions. If a particular solution is desired, then we must prescribe additional restrictions, or "side conditions."

The upshot of all this is that with a partial differential equation, we must also be told where the equation applies and what kind of functions are acceptable as solutions. This information is usually supplied from the application where the equation originated. However, there are important cases when the "side conditions" are not clear from the application, and have to be determined by studying the equation. They are then used to determine "meaningful" situations in the application.

Needless to say, the number of partial differential equations (and systems of equations) that can be dreamt up is infinite. The number of equations arising in applications is not much smaller. To complicate matters, experience has shown us that a slight modification of an equation (such as the change in sign of a term) may cause solutions to be completely different in nature, with entirely different methods required for solving them. It should come as no surprise, therefore, that as yet we are nowhere near a systematic treatment of partial differential equations. At best, the present state of knowledge can be described as a conglomeration of particular methods (the word "tricks" may even be more appropriate) which work in special cases. Thus, any treatment of partial differential equations, no matter how extensive, must necessarily restrict itself to a relatively small area of the subject.

We have chosen to deal with *linear* partial differential equations primarily because they are the easiest to deal with. The most general linear partial differential equation involving one unknown function $u(x_1, \ldots, x_n)$ can be written in the form

$$\sum_{\mu_1 + \cdots + \mu_n \leq m} a_{\mu_1, \ldots, \mu_n}(x_1, \ldots, x_n) \frac{\partial^{\mu_1 + \cdots + \mu_n} u}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} = f(x_1, \ldots, x_n) \qquad (1\text{-}3)$$

where summation is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$, and the a's and f are given functions. (Since we have not as yet defined what we mean by a linear equation, you might as well take Eq. (1-3) to be the definition.)

One look at Eq. (1-3) should be sufficient to discourage anyone from studying partial differential equations. (If it does not accomplish this effect, I shall do better later on.) However, once we have survived the initial impact, we see that a bit of shorthand will do a lot of good. For instance, if we let μ stand for the *multi-index* $(\mu_1, \ldots, \mu_n)$ with *norm* $|\mu| = \mu_1 + \cdots + \mu_n$, and let x stand for the vector $(x_1, \ldots, x_n)$ and write

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}$$

then Eq. (1-3) becomes

$$\sum_{|\mu| \leq m} a_\mu(x) \, D^\mu u(x) = f(x) \tag{1-4}$$

which looks much better.

Let us examine Eq. (1-4) a little more closely. The left-hand side consists of a sum of terms, each of which is a product of a coefficient and a derivative of u. We may consider it as a *differential operator* A acting on u. We can then write Eq. (1-4) more simply as

$$Au = f \tag{1-5}$$

where
$$A = \sum_{|\mu| \leq m} a_\mu(x) \, D^\mu \tag{1-6}$$

The operator A is called linear because

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2 \tag{1-7}$$

holds for all functions u_1, u_2 and all numbers α_1, α_2. Equation (1-5) is called linear because the operator A is linear.

Now what do we mean by a solution of Eq. (1-4)? Since derivatives up to and including those of order m are involved, it seems quite natural to require that these derivatives exist and are continuous, and when they are substituted into Eq. (1-4) the equality holds. We take this as our present definition. Later on, we shall find it convenient, if not essential, to modify this definition quite drastically.

Where do we want Eq. (1-4) to hold? Obviously, it should hold in some subset Ω of $(x_1, \ldots, x_n)$ space. This subset has to be specified. Of course, $u(x)$ has to be defined in a neighborhood of each point of Ω in order that the appropriate derivatives be defined. Since we want our solutions to have continuous derivatives up to order m in Ω, we shall give this set of functions a name. We denote the n-dimensional coordinate space by E^n.

Definition 1-1 Let Ω be a set in E^n. We let $C^m(\Omega)$ denote the set of all functions defined in a neighborhood of each point of Ω, and having all derivatives of order $\leq m$ continuous in Ω. If a function u is in $C^m(\Omega)$ for each m, then it is called infinitely differentiable and said to be in $C^\infty(\Omega)$, i.e.,

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

For $m = 0$, we write $C(\Omega) = C^0(\Omega)$. This is the set of functions continuous in Ω.

1-2 EQUATIONS WITHOUT SOLUTIONS

The first question that might be asked concerning Eq. (1-4) is whether or not it has a solution in a given set Ω. To make the environment as conducive as possible, let us be willing to take Ω as the sphere Σ_r consisting of those points $(x_1, \ldots, x_n)$ of E^n satisfying

$$x_1^2 + \cdots + x_n^2 < r^2$$

where r is some positive number. (The reason for calling Σ_r a sphere should be evident.) Let us even be willing to assume that the function f and the coefficients of A are infinitely differentiable in Σ_r (i.e., they are in $C^\infty(\Sigma_r)$). Under such circumstances one might reasonably expect a solution of Eq. (1-4) to be guaranteed. Unfortunately, this is decidedly not the case. A simple example was discovered by H. Lewy (1957) (pronounced Layvee), and a study of it is instructive.

The setting is three-dimensional space and we denote the coordinates by x, y, t (we save the letter z for another quantity). The equation is simple to write down:

$$u_x + iu_y + 2(ix - y)u_t = f(x, y, t) \tag{1-8}$$

A word of explanation is in order. The coefficients of this equation are complex-valued, while it was hitherto tacitly assumed that the functions and coefficients considered were real-valued. The following considerations will clarify the matter.

Suppose we allow f to be complex-valued in the sense that there are two bona fide, real-valued functions $f_1(x, y, t)$ and $f_2(x, y, t)$, such that $f = f_1 + if_2$. It is to be understood that there need not be any connection between the two functions f_1 and f_2. We assume the same for any solution u. Then Eq. (1-8) is equivalent to the system

$$u_{1x} - u_{2y} - 2xu_{2t} - 2yu_{1t} = f_1 \tag{1-9}$$

$$u_{2x} + u_{1y} + 2xu_{1t} - 2yu_{2t} = f_2 \tag{1-10}$$

which involves only real functions. It is a system of two equations in two unknowns. Thus, Eq. (1-8) is just a short way of writing Eqs. (1-9) and (1-10). The fact that Eq. (1-8) is a system, is not a factor in its lack of solutions. We shall also exhibit a single equation with real f and real coefficients which has no solution.

Now back to Eq. (1-8). To simplify it, we introduce the complex variable $z = x + iy$. Then $u(x, y, t)$ is a function of z and t. It is an analytic function of z only if it satisfies the Cauchy–Riemann equations

$$u_{1x} = u_{2y} \qquad u_{1y} = -u_{2x} \tag{1-11}$$

or their abbreviated form

$$u_x + iu_y = 0 \tag{1-12}$$

To abbreviate even further, set

$$2u_z = u_x - iu_y \qquad 2u_{\bar{z}} = u_x + iu_y \tag{1-13}$$

Then Eq. (1-12) becomes

$$u_{\bar{z}} = 0 \tag{1-14}$$

while Eq. (1-8) becomes

$$u_{\bar{z}} + izu_t = \tfrac{1}{2}f \tag{1-15}$$

Now let Ω be the set $x^2 + y^2 < a$, $|t| < b$, where a and b are any fixed positive numbers. We shall show that there is an $f \in C^\infty(\Omega)$ such that Eq. (1-8) has no solution in $C^1(\Omega)$. Since a and b are arbitrary, it will follow that Eq. (1-8) does not have a solution in Σ_r for any $r > 0$.

To carry out our proof, let $\psi(\sigma, \tau)$ be a continuously differentiable complex-valued function of two real variables σ, τ which vanishes outside the rectangle $0 < \sigma < a$, $|\tau| < b$. Set

$$\varphi(x, y, t) = \psi(\rho, t) \qquad \rho = x^2 + y^2$$

Note that φ has continuous derivatives in x, y, t space and vanishes outside Ω. By the chain rule, we have

$$\varphi_z(x, y, t) = \bar{z}\psi_\rho(\rho, t) \tag{1-16}$$

Now suppose there were a solution u of Eq. (1-8) in Ω. Then,

$$\iiint_\Omega (u_{\bar{z}} + izu_t)\bar{\varphi}\, dx\, dy\, dt = \frac{1}{2}\iiint_\Omega f\bar{\varphi}\, dx\, dy\, dt$$

where the bar denotes complex conjugation. Integrating the left-hand integral by parts (see Sec. 1–3), we have

$$-\iiint_\Omega u\overline{(\varphi_z - i\bar{z}\varphi_t)}\, dx\, dy\, dt = \frac{1}{2}\iiint_\Omega f\bar{\varphi}\, dx\, dy\, dt$$

(There are no boundary integrals because φ vanishes on the boundary of Ω). By Eq. (1-16) this becomes

$$-\iiint_\Omega zu\overline{(\psi_\rho - i\psi_t)}\, dx\, dy\, dt = \frac{1}{2}\iiint_\Omega f\bar{\varphi}\, dx\, dy\, dt \tag{1-17}$$

We now introduce coordinates ρ, θ in place of x and y, where

$$\tan\theta = \frac{y}{x}$$

Noting that $2d\rho\, d\theta = dx\, dy$, we see that Eq. (1-17) becomes

$$-\int_{-b}^{b}\int_0^{2\pi}\int_0^a zu\overline{(\psi_\rho - i\psi_t)}\, d\rho\, d\theta\, dt = \frac{1}{2}\int_{-b}^{b}\int_0^{2\pi}\int_0^a f\bar{\psi}\, d\rho\, d\theta\, dt \tag{1-18}$$

We now set

$$U(\rho, t) = \int_0^{2\pi} zu\, d\theta \tag{1-19}$$

and assume that f does not depend on θ. Since ψ also does not depend on θ, we have

$$-\int_{-b}^{b}\int_0^a U\overline{(\psi_\rho - i\psi_t)}\, d\rho\, dt = \pi\int_{-b}^{b}\int_0^a f\bar{\psi}\, d\rho\, dt$$

We now integrate the left-hand side by parts, obtaining

$$\int_{-b}^{b}\int_{0}^{a}(U_\rho + iU_t - \pi f)\bar{\psi}\, d\rho\, dt = 0$$

The next step is to note that ψ was any continuously differentiable function which vanished outside of $0 < \rho < a, |t| < b$. It follows from well-known arguments (see Sec. 1–3) that

$$U_\rho + iU_t = \pi f \qquad 0 < \rho < a \qquad |t| < b \tag{1-20}$$

Next take $f = g'(t)$, where g is a smooth, real-valued function of t alone, and set

$$V(\rho, t) = U + \pi i g \tag{1-21}$$

Then
$$V_\rho + iV_t = 0 \qquad 0 < \rho < a \qquad |t| < b$$

and hence V is an analytic function of $\rho + it$ on this set. Since $u(x, y, t)$ is continuous on $0 \le \rho < a, |t| < b$, so is $U(\rho, t)$. Moreover, $U(0, t) = 0$ by Eq. (1-19). Thus

$$\text{Re } V(0, t) = 0 \qquad |t| < b \tag{1-22}$$

Since V is analytic in $0 < \rho < a, |t| < b$, and its real part vanishes for $\rho = 0$, we know that we can continue V analytically across the line $\rho = 0$ (see any good book on complex variables). In particular, $V(0, t)$ is an analytic function of t in $|t| < b$ (in the sense of power series). But $V(0, t) = \pi i g(t)$. Thus, we have shown that in order for Eq. (1-8) to have a solution when f depends on t alone, it is necessary that f be an analytic function of t. If we take, for example

$$g(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0 \end{cases} \tag{1-23}$$

then f has continuous derivatives of all orders, but is not analytic in any neighborhood of $t = 0$. Hence, Eq. (1-8) can have no solution for such an f.

Now we can give an example of a "real" equation without solutions. Let Au stand for the left-hand side of Eq. (1-15). Let $\bar{A}$ represent the operator obtained from A by taking the complex conjugate of all of the coefficients in A.

Then
$$A\bar{A}u = \tfrac{1}{4}(u_{xx} + u_{yy}) + (xu_{yt} - yu_{xt}) + (x^2 + y^2)u_{tt} - iu_t \tag{1-24}$$

This, unfortunately, does not quite make the grade because of the last term. But we do have

$$A\bar{A} = B - i\frac{\partial}{\partial t}$$

where B is a linear operator with real coefficients.

Thus
$$A\bar{A}(\overline{A\bar{A}}) = \left(B - i\frac{\partial}{\partial t}\right)\left(B + i\frac{\partial}{\partial t}\right) = B^2 + \frac{\partial^2}{\partial t^2}$$

Now, I claim that the equation

$$B^2 u + u_{tt} = f \tag{1-25}$$

cannot have a solution when $f = g'$ and g is given by Eq. (1-23). For if u were a solution of Eq. (1-25), then $v = \frac{1}{2}\bar{A}(\overline{A\bar{A}})u$ would be a solution of Eq. (1-8), contradicting our previous result. This example was given by F. Treves (1962).

It might be noted that Eq. (1-25) would be much harder to deal with directly. The fact that we were allowed to use complex-valued functions brought about a great savings. This is true in many other situations in the study of partial differential equations.

1-3 INTEGRATION BY PARTS

In Sec. 1-2 we employed an elementary but very useful technique, which we will review here for the benefit of anyone who is a bit rusty. It is integration by parts. Let Ω be an open, connected set (domain) in E^n with a piecewise smooth boundary. This means that the boundary $\partial\Omega$ of Ω consists of a finite number of surfaces each of which can be expressed in the form

$$x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

for some j, with the function h having continuous first derivatives. The closure $\bar{\Omega}$ of Ω is the union of Ω and its boundary $\partial\Omega$. Assume that Ω is bounded, i.e., that it is contained in some Σ_R for R sufficiently large. If $f \in C^1(\bar{\Omega})$, then

$$\int_\Omega \frac{\partial f}{\partial x_k}\, dx = \int_{\partial\Omega} f\gamma_k\, d\sigma \qquad 1 \le k \le n \tag{1-26}$$

where $dx = dx_1 \cdots dx_n$, γ_k is the cosine of the angle between the x_k-axis and the outward normal to $\partial\Omega$, and $d\sigma$ is the surface element on $\partial\Omega$. (Note that we use only one integral sign for a volume integral; it would not be easy to write n of them.) Equation (1-26) has many names attached to it, including Gauss, Green, Stokes, divergence, etc. For a proof we can refer to any good book on advanced calculus, e.g., Spivak (1965).

Now suppose u and v have continuous derivatives in $\bar{\Omega}$ and their product vanishes on $\partial\Omega$. Then, by Eq. (1-26), we have

$$\int_\Omega u \frac{\partial v}{\partial x_k}\, dx = -\int_\Omega \frac{\partial u}{\partial x_k} v\, dx \qquad 1 \le k \le n \tag{1-27}$$

This is the formula employed in Sec. 1-2. It is a very convenient one, since it allows us to "throw" derivatives from one function to another. It is so convenient that the first general rule for all people studying partial differential equations is: when you do not know what to do next, integrate by parts.

There is one feature of Eq. (1-27) which appears harmless, but which has done more to fill mental institutions with partial differential equations people than any other single factor, namely, the minus sign. However, there is a way of avoiding it. The method is as follows. As agreed before, we can allow complex-valued functions provided we understand that there need not be any connection between their real