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Surveys on Recent Developments in Algebraic Geometry

Bootcamp for the
2015 Summer Research Institute
on Algebraic Geometry
July 6–10, 2015
University of Utah, Salt Lake City, Utah

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Surveys on Recent Developments in Algebraic Geometry

Preface

The algebraic geometry community has a tradition of running a research institute every ten years. This important conference convenes a majority of the practitioners in the world to overview the developments of the past decade and to outline the most fundamental and far-reaching problems for the next. Previous institutes have included Woods Hole (1964), Arcata (1974), Bowdoin (1985), Santa Cruz (1995) and Seattle (2005). This past Algebraic Geometry Summer Institute took place at the University of Utah, in Salt Lake City, on July 13–31, 2015.

Increasingly, algebraic geometry has become very diverse and technically demanding. It is daunting for graduate students and postdocs to master the techniques and wide-range of activities in the field. In response, the community has been running a "Bootcamp" in the week preceding the institute with the aim of familiarizing the participants with a broad-range of developments in algebraic geometry in an informal setting. In 2015, from July 6 to 10, the Bootcamp took place at the University of Utah, Salt Lake City. Following tradition, the 150 graduate student participants were mentored by 15 postdocs and young assistant professors, helping to create a vibrant and informal atmosphere and allowing young researchers to form support networks. The mentors introduced the graduate students and postdocs to exciting new developments in the Minimal Model Program, Hodge theory, perfectoid spaces, positive characteristic techniques, Boij-Söderberg theory, p-adic Hodge theory, Bridgeland stability, tropical geometry and many more topics at the cutting edge of the field.

Activities at the Bootcamp included morning lectures given by the mentors, followed by afternoon working sessions. In this volume, in an attempt to make these excellent expositions more widely available, we have collected 10 survey papers that grew out of the lectures. Each paper discusses a different subfield of algebraic geometry that has seen significant progress in the last decade. The papers preserve the informal tone of the lectures and strive to be accessible to graduate students who have basic familiarity with algebraic geometry. They also contain many illuminating examples and open problems. We expect these surveys will become invaluable resources, not only for graduate students and postdocs, but also senior researchers starting along new directions.

We now summarize the contents of this volume in greater detail.

Higher dimensional birational geometry. The Minimal Model Program was initiated in the 1980s by Mori as a way of extending classification theorems for surfaces to higher dimensional varieties. In the last decade, several of the central conjectures in the field have been resolved. The fundamental and influential work of Birkar, Cascini, Hacon and McKernan showed the existence of minimal models for varieties of general type, and proved the finite generation of the canonical ring.

The consequences have ranged from the resolution of classical conjectures such as the Sarkisov program to the construction of new moduli spaces. The successes have also led to attempts at extending the theory to other contexts such as Kähler manifolds or characteristic p birational geometry. The paper *A snapshot of the Minimal Model Program* by Brian Lehmann describes the most important developments of the decade and gives a list of open problems and conjectures. The paper *Positive characteristic algebraic geometry* by Patakfalvi, Schwede and Tucker gives a detailed introduction to the use of the Frobenius morphism for defining birational and singularity invariants in characteristic p. This approach has led to tremendous advances including an extension of the Minimal Model Program to threefolds in characteristic $p > 5$. The paper contains many illustrative and instructional examples and exercises to familiarize the reader with the characteristic p techniques.

Progress on moduli spaces. Moduli spaces have been at the center of algebraic geometry and its applications to other areas of mathematics. Moduli spaces of curves, moduli spaces of abelian varieties and moduli spaces of sheaves appear in many guises in mathematics and have applications to number theory, combinatorics, mathematical physics and topology. In the last decade, there has been significant development in constructing new compactifications and new birational models of important moduli spaces. The developments were motivated in part by evolution in the Minimal Model Program and new techniques coming from derived categories and Bridgeland stability. As a result, our understanding of classical moduli spaces such as the moduli space of principally polarized abelian varieties and moduli spaces of Gieseker semistable sheaves on surfaces has vastly improved. The paper *The geometry of the moduli space of curves and abelian varieties* by Tommasi gives an overview of this progress. Tommasi's paper is notable for an accessible account of toroidal compactifications of the moduli spaces of abelian varieties. The paper *Birational geometry of moduli spaces of sheaves and Bridgeland stability* by Jack Huizenga gives a masterful introduction to the geometry of moduli spaces of sheaves on surfaces and Bridgeland stability conditions using the projective plane as a motivating example. In recent years, Bridgeland stability has revolutionized our understanding of the birational geometry of moduli spaces of sheaves on surfaces. Arcara, Bayer, Bertram, Coskun, Huizenga, Macrì and others have computed the ample and effective cones of moduli spaces of sheaves on certain surfaces and have given many geometric applications. Huizenga's paper also clearly illustrates the interactions between developments in the Minimal Model Program and moduli spaces.

New applications of moduli spaces and connections to dynamics. There has been significant expansion in the applications of moduli theory and in the interactions of moduli spaces with dynamics. Inspired by string theory, Gromov-Witten theory revolutionized classical enumerative geometry in the 1990s and early 2000s. In the last decade, the theory has matured and found new applications to the tautological ring of the moduli space of curves. Maulik, Nekrasov, Okounkov, Pandharipande conjectured the equivalence of various curve counting theories such as the Gromov-Witten and Donaldson-Thomas correspondence. Pandharipande, Pixton and others have resolved some of these conjectures. The paper *Gromov-Witten theory: From curve counts to string theory* by Clader, gives a succinct account of the vast advances that have taken place in Gromov-Witten theory in the last decade.

Clader masterfully summarizes the correspondence between Gromov-Witten and Donaldson-Thomas theory and applications of Gromov-Witten theory to the tautological ring of the moduli space of curves. She concludes with a set of central open problems in the field. The paper *Teichmüller dynamics in the eyes of an algebraic geometer* by Chen, introduces algebraic geometers to recent developments in Teichmüller dynamics following the fundamental work of Eskin, Kontsevich, Mirzakhani and Zorich. In his beautifully illustrated survey, Chen makes many analytic and dynamical concepts accessible to algebraic geometers. Chen emphasizes applications of the theory to the geometry of the Deligne-Mumford moduli spaces of curves and degenerations of abelian differentials.

Rationality of varieties. The last few years have seen an explosion in the study of rationality of varieties. Voisin introduced a new deformation technique based on the Chow theoretic decomposition of the diagonal to show that very general quartic double solids are not stably rational. Her approach was extended by Colliot-Thélène and Pirutka, and applied by them and many others such as Totaro, Hassett and Tschinkel to resolve long standing questions of rationality and stable rationality. In addition, Kuznetsov and others have suggested using the derived category and orthogonal decompositions to obtain new invariants for rationality. The survey *Unramified cohomology, derived categories and rationality* by Auel and Bernardara, gives a comprehensive introduction to these novel ideas and explores the interconnections between the two developments.

Hodge theory and degenerations. Hodge theory carries subtle transcendental information about the geometry of complex varieties. In the last decade, new insights have allowed a better understanding of degenerations of Hodge structures and the corresponding degenerations of varieties. The paper *Degenerations of Hodge structure* by Robles is an introduction to recent developments in Hodge theory, most notably to the classification of certain degenerations of Hodge structure pioneered by Green, Griffiths, Kerr, Robles and others.

Syzygies and cohomology tables. Betti tables of resolutions and cohomology tables of vector bundles are fundamental objects of study in commutative algebra and algebraic geometry. Eisenbud and Schryer's solution of the Boij-Söderberg conjecture and the resulting description of the cones of these tables have reshaped the theory in the last decade. The paper *Questions about Boij-Söderberg theory* by Erman and Sam surveys the developments and poses many fascinating and accessible further open problems.

Homotopy methods in algebraic geometry. The motivic, or $\mathbb{A}^1$ homotopy theory, introduced by Morel and Voevodsky, lies at the heart of recent progress, such as the classification of vector bundles on smooth complex affine varieties by Asok and Fasel. The paper *A primer for unstable motivic homotopy theory* by Antieau and Elmanto, gives an accessible introduction to this technical theory. Many key examples, useful exercises, enticing open problems and extensive references make this paper an indispensable reference for beginning practitioners of the subject.

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A snapshot of the Minimal Model Program

Brian Lehmann

ABSTRACT. We briefly review the main goals of the minimal model program. We then discuss which results are known unconditionally, which are known conditionally, and which are still open.

1. Introduction

This survey paper reviews the main goals and results of the Minimal Model Program (henceforth MMP). The paper has three parts. In Section 2, we give a very gentle introduction to the main ideas and conjectures of the MMP. We emphasize why the results are useful for many different areas of algebraic geometry.

In Sections 3-6, we take a "snapshot" of the MMP: we describe which results are currently known unconditionally, which are known conditionally, and which are wide open. We aim to state these results precisely, but in a manner which is as useful as possible to as wide a range of mathematicians as possible. Accordingly, this paper becomes more and more technical as we go. The hope is this paper will be helpful for mathematicians looking to apply the results of the MMP in their research.

In Section 7, we briefly overview current directions of research which use the MMP. Since the foundations of the MMP are discussed previously, this section focuses on applications. The choice of topics is not comprehensive and is idiosyncratically based on my own knowledge.

These notes are *not* intended to be a technical introduction to the MMP. There are many good introductions to the techniques of the MMP already: [**KM98**], [**Mat02**], [**HK10**], [**Kol13b**], and many others. These notes are also *not* intended to be a historical introduction. We will focus solely on the most recent results which are related to Principle 2.2: the existence of minimal models, termination of flips, and the abundance conjecture. Thus we will not cover in any length the many spectacular technical developments required as background. In particular, we will unfortunately omit most of the foundational results from the 1980's due to Kawamata, Kollár, Mori, Reid, Shokurov, and many others. We will also not cover the analytic side of the picture in any depth. To partially amend for this decision, we give a fairly complete list of references at the end.

Throughout we will work over $\mathbb{C}$ unless otherwise specified. Varieties are irreducible and reduced.

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2. Main idea of the MMP

WARNING 2.1. This entire section is filled with inaccuracies, imprecisions, oversimplifications, and outright falsehoods. The few terms which may be new to a general audience will be defined rigorously in the next section.

I will focus on the guiding principle:

PRINCIPLE 2.2. Let X be a smooth projective variety over a field. The geometry and arithmetic of X are governed by the "positivity" of the canonical bundle $\omega_X := \bigwedge^{\dim X} \Omega^1_X$.

We will participate in the traditional abuse of notation by letting K_X denote any Cartier divisor satisfying $\mathcal{O}_X(K_X) \simeq \omega_X$. Such a divisor is only unique up to linear equivalence, but since our statements are all linear-equivalence invariant this abuse is harmless in practice.

We first work out the case of dimension 1. Let C be a smooth integral projective curve with canonical bundle $\omega_C = \Omega_C$. The central feature of the curve is its genus:

$$g_C = \dim H^0(C, \Omega_C).$$

(Hodge theory shows that this coheres with the classical definition via Betti numbers.) By Riemann-Roch and Serre duality this is equivalent to saying that:

$$\deg(\Omega_C) = 2g_C - 2.$$

As is well-known, curves split into three categories based upon their genus or curvature. In line with Principle 2.2, it is most natural to describe this trichotomy in terms of the degree of the canonical bundle – only then do we see why the trichotomy is the right one.

$\deg(K_C)$	< 0	$= 0$	> 0
g_C	0	1	≥ 2
examples	$\mathbb{P}^1$	elliptic curves	smooth plane curves of degree > 3
universal cover/$\mathbb{C}$	$\mathbb{P}^1$	$\mathbb{C}$	$\mathbb{H}$
automorphisms	PGL_2	$\approx$ itself	finite
rational points over # field	dense after deg 2 ext	dense and thin after ext	finite

It would be very nice to have a similar trichotomy in higher dimensions. Of course this is too optimistic – a complex manifold of higher dimensions can have different curvatures in different directions – but we will soon see that there is some hope.

For arbitrary varieties X, we first need to decide what properties of K_X best correspond to the conditions $\deg(K_C) < 0$, $= 0$, > 0 for curves. Ample divisors are the most natural generalization of "positive degree" divisors for curves (and dually for antiampleness and "negative degree"). In the setting of the MMP, the best analogue of "degree 0" turns out to be the condition that K_X is torsion – some multiple of K_X is linearly equivalent to the 0 divisor. When K_X is ample, torsion, or antiample we say that our variety has "pure type".

With these changes, the trichotomy we found for curves seems to hold up well in higher dimensions. However, many of statements are still only conjectural – these will be designated with a question mark in the table below.

K_X	antiample	$\sim_{\mathbb{Q}} 0$	ample
examples	$\mathbb{P}^n$ Fanos	abelian varieties hyperkählers Calabi-Yaus	high degree hypersurfaces in $\mathbb{P}^n$
fundamental group/$\mathbb{C}$	1	almost abelian	??
rational curves on $X/\mathbb{C}$	dense	contained in countable union of proper subsets	not dense?
rational points over # field	potentially dense?	??	not dense?

While this conceptual picture is very appealing, at first glance it seems to only address a very limited collection of varieties. The main conjecture of the MMP is that *any* variety admits a "decomposition" into these varieties of pure types: at least after passing to a birational model, we can find a fibration with pure type fibers.

CONJECTURE 2.3 (Guiding conjecture of the MMP). *Any smooth projective variety X admits either:*

(i) a birational model $\psi : X \dashrightarrow X'$ and a morphism $\pi : X' \to Z$ with connected fibers to a variety of smaller dimension such that the general fiber F of π has K_F antiample, or

(ii) a birational model $\psi : X \dashrightarrow X'$ and a morphism $\pi : X' \to Z$ with connected fibers to a variety of smaller dimension such that the general fiber F of π has K_F torsion, or

(iii) a birational model $\psi : X \dashrightarrow X'$ and a birational morphism $\pi : X' \to Z$ such that K_Z is ample.

We will refer to the outcomes respectively as cases (i), (ii), (iii). It is clear why Conjecture 2.3 is so powerful – it suggests that we can leverage results for pure type varieties to study any variety via a suitable fibration.

Historically, this conjecture has its roots in the Kodaira-Enriques classification of surfaces, which categorizes birational equivalence classes of surfaces exactly according to the ability to find a morphism with fibers of a given pure type. (The fact that the birational map ψ may not be defined everywhere is a new feature in higher dimensions.)

REMARK 2.4. Even when X is a smooth surface, the varieties predicted by Conjecture 2.3 may have certain "mild" singularities. In this section we will ignore singularities completely, but the reader should remember they are there.

Implicit in the statement of Conjecture 2.3 is that the three cases have a hierarchy ordered by negativity: we look for an antiample fibration, then (failing to find any) a torsion fibration, then (failing that too) we expect to be in case (iii).

The justification is that it is quite easy to construct subvarieties with ample canonical divisor – for example, take complete intersections of sufficiently positive very ample divisors. The "most special" subvarieties are those with antiample canonical divisor, and we should look for these subvarieties first. (As we will see soon, this hierarchy is more properly motivated by the birational properties of K_X.)

The apparent asymmetry of case (iii) is also justified by this logic. Every variety admits many rational maps with fibers of general type, and the existence of such maps tells us essentially nothing about the variety. In contrast, Conjecture 2.3 has real geometric consequences.

REMARK 2.5. Another common perspective on the MMP is that it identifies a "distinguished set" of representatives of each fixed birational equivalence class of varieties. In dimension 2, the Kodaira-Enriques classification identifies a unique smooth birational model of any surface and we obtain a birational classification of surfaces. In higher dimensions, the analogous constructions are not unique, and so this perspective is slightly less useful.

Conjecture 2.3 suggests the following questions:

(a) How can we identify the target Z, or equivalently, the rational map $\phi = \pi \circ \psi : X \dashrightarrow Z$?
(b) How can we identify the rational map ψ and the birational model X'? What properties of X' distinguish it as the "right" birational model?
(c) How can we predict the case (i), (ii), (iii) of X based on the geometry of K_X?

We will answer these questions in the following subsections.

2.1. Canonical models. We first turn to Question (a): how to identify the variety Z? In other words, how can we naturally choose a rational map $\phi : X \dashrightarrow Z$ which captures the essential geometric features of X? For now we will focus on cases (ii) and (iii); case (i) is somewhat different.

Rational maps are constructed from sections of line bundles on X. For arbitrary varieties we only really have access to one line bundle: the canonical bundle. Furthermore, the canonical bundle encodes fundamental information about the curvature of our variety X. Thus it is no surprise that our "canonical map" ϕ should be constructed from the canonical divisor K_X.

A fundamental principle of birational geometry is that the geometry of a divisor L is best reflected not by sections of L but by working with all multiples of L simultaneously. We obtain access to this richer structure by the following seminal theorem of [**BCHM10**].

THEOREM 2.6 ([**BCHM10**] Corollary 1.1.2). *Let X be a smooth projective variety. Then the pluricanonical ring of sections*

$$R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, mK_X)$$

is finitely generated.

Suppose that some multiple of K_X has sections (which should happen – and can only happen – in cases (ii) and (iii)). Since the pluricanonical ring is finitely generated, we can take its Proj.

DEFINITION 2.7. Suppose that some multiple of K_X has sections. The canonical model of X is defined to be $\operatorname{Proj} R(X, K_X)$.