

Combinatorial Geometry *in the Plane*

by

H. HADWIGER

and

H. DEBRUNNER

Translated by Victor Klee
University of Washington

Combinatorial Geometry in the Plane

By Hugo Hadwiger and Hans Debrunner
University of Bern

WITH A NEW CHAPTER AND
OTHER ADDITIONAL MATERIAL
SUPPLIED BY THE TRANSLATOR

HOLT, RINEHART AND WINSTON
New York • Chicago • San Francisco
Toronto • London

***Combinatorial Geometry
in the Plane***

**Athena
Series**



SELECTED TOPICS IN MATHEMATICS

Edwin Hewitt, *Editor*

AMBROSE, LAZEROWITZ: LOGIC: THE THEORY OF FORMAL INFERENCE

BELLMAN: A BRIEF INTRODUCTION TO THETA FUNCTIONS

BELLMAN: PERTURBATION TECHNIQUES IN MATHEMATICS, PHYSICS,
AND ENGINEERING

CARDANO: THE BOOK ON GAMES OF CHANCE

ERDÉLYI: OPERATIONAL CALCULUS AND GENERALIZED FUNCTIONS

FICKEN: THE SIMPLEX METHOD OF LINEAR PROGRAMMING

HADWIGER, DEBRUNNER, AND KLEE: COMBINATORIAL GEOMETRY
IN THE PLANE

HEINS: SELECTED TOPICS IN THE CLASSICAL THEORY OF FUNCTIONS
OF A COMPLEX VARIABLE

HIRSCHMAN: INFINITE SERIES

HOCHSTADT: SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS

JANS: RINGS AND HOMOLOGY

KAZARINOFF: ANALYTIC INEQUALITIES

WILLIAMSON: LEBESGUE INTEGRATION

Preface

On both high school and collegiate levels, the content and approach of our mathematics curriculum have changed materially in recent years. Though many pedagogic improvements have resulted, three important problems are unsettled. One of these, widely recognized by the professional committees that have sparked the curriculum change, is "What should be done about geometry?" Powerfully influenced by Euclid's example of over two thousand years ago, some mathematicians feel that geometry is still the ideal subject for introducing the student to rigorous thought. Others argue that the logical structure of geometry is much too complicated for this purpose, that Euclid had no choice because geometry was *the* mathematics of his day, and that in any case the traditional axiomatic approach to geometry is inefficient.

The other two questions are "How should we encourage the student to develop his mathematical intuition, and how help him to understand the nature of mathematical research?" It seems to us that these two questions are closely related and have a common answer. We should expose the student to some material that has strong intuitive appeal, is currently of research interest to professional mathematicians, and in which the student himself may discover interesting problems that even the experts are unable to solve. There are two areas in mathematics that clearly conform to this description. One is elementary number theory and the other is the sort of geometry that forms the content of this book, largely associated with the notion of convexity. We submit the present book as a tentative answer to all three of the questions mentioned above. Evidence favoring this sort of solution is provided by the use in Russian schools of several works on convex bodies,

especially the book by Yaglom and Boltyanskii [114] which is an excellent companion volume for the present one.*

In all essentials, the translation adheres closely to the original text. At the authors' invitation, the translator has added a number of references and also an eleventh chapter that surveys recent developments in the field. This is based partly on his joint paper with Ludwig Danzer and Branko Grünbaum, whose collaboration was also useful to the authors in preparing the original text.

Seattle, Washington
October, 1963

V.K.

*The book by Yaglom and Boltyanskii has been translated into both German and English, the latter edition being published by Holt, Rinehart and Winston, Inc. The German edition of the present book was based on the article "Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene" by H. Hadwiger in *L'Enseignement Mathématique* (2), 1, 56-89 (1955), translated into French by J. Chatelet, *ibid.* 3, 35-70 (1957).

German Language Edition:

*Kombinatorische Geometrie
in der Ebene*, Monographies
de L'Enseignement Mathematique,
No. 2, Geneva.

ENGLISH TRANSLATION

Copyright © 1964 by

Holt, Rinehart and Winston, Inc.

Library of Congress Catalog Card Number: 64-10297

24979-0114

Printed in the United States of America

All Rights Reserved

Contents

Preface	V
Introduction	1
Part I	3
1. Incidence of Points, Lines, and Circles	3
2. Integral Distances. Commensurable Angles	4
3. Hull Formation. Separability	6
4. Helly's Theorem. Transversal Problems for Ovals	7
5. Covering Problems	13
6. Point Set Geometry and Convexity	20
7. Realization of Distances	21
8. Simple Paradoxes Involving Point Sets	25
9. Pure Combinatorics. Graphs	27
10. Additional Theorems of Helly Type	31
11. Further Development of Combinatorial Geometry (by the translator)	40
Part II: Proofs	57
Bibliography	99
Index	111

Introduction

There are various mathematical subjects in which elementary exercises lead at once to more advanced and partially unsolved problems, so that the simplest matters of school mathematics are closely tied to those that are of scientific interest and are studied by specialists. A key point in this is that the two professional levels are not, as is usual, separated from each other by highly developed advanced theories and stratified scales of ideas.

Such a subject is combinatorial geometry, which has an especially simple character when restricted to the plane. Its problems are directly connected with the basic ideas of elementary plane geometry and are based on various primitive relations and processes, such as those of inclusion, intersection, decomposition, and so forth, and on the combinatorial possibilities associated with these relations and processes.

The subject is related to combinatorial topology; however, genuine topological considerations remain well in the background, and the difficulties are those of elementary geometry. As was more fully explained by H. Hopf [47], there is a certain reciprocal relationship between the metric and topological viewpoints in combinatorial geometry.

The gathering of numerous special problems that we have undertaken is not, however, totally restricted to the methods of combinatorial geometry; these form only a tiny nucleus of a complex of questions that has exerted a special attraction because of the simple and basic nature of its subject matter, and the purely combinatorial aspect of the necessary inferences.

How a person familiar only with elementary concepts can pose problems so as to develop this taste, and accustom himself to a change that leads from the methods and topics of the familiar classical domain to those of a more currently oriented research area with exciting new possibilities — this will be brought home to the reader by the examples collected here.

Except for basic material from elementary geometry and the theory of real numbers, little is needed in the way of previous knowledge; a certain familiarity with set-theoretical reasoning is useful, and the notion of a plane point set is important. Most of the notation will be briefly explained.

In Part I, selected theorems are assembled, arranged into groups of related propositions without proof but with rather detailed comments and with references to the literature. The proofs, often only briefly indicated, follow in Part II. Thus the reader will have the opportunity to practice the search for and execution of his own ideas of proof. Through the numerous

references, especially interested readers may also find their way to the current research literature and may even pursue the unsolved problems that are suggested.

With these selected special problems we hope to stimulate an intensive study of the fascinating questions of combinatorial geometry and to bring into active being the close contact that exists between school mathematics and scientific research in this domain.

Part I

1. Incidence of Points, Lines, and Circles

The propositions of this first small group deal with incidence relations among points, lines, and circles, and thus pertain to combinatorial elementary geometry.

1. If a finite set of points is such that on the line determined by any two of the points there is always a third point of the set, then all the points lie on a single line.

This theorem was conjectured in 1893 by J. J. Sylvester [102]. A short proof due to T. Gallai (Grünwald) is given by N. G. de Bruijn–P. Erdős [9], where the result appears as a corollary of a purely combinatorial theorem. For further proofs, generalizations, and variants see P. Erdős [16], H. S. M. Coxeter [10], G. A. Dirac [13], and Th. Motzkin [76].

2. If a finite set of lines is such that through the intersection point of any two of the lines there always passes a third line of the set, then all the lines pass through a single point.

Propositions **1** and **2** are no longer true if the sets of points or lines are infinite. This is demonstrated for both theorems simultaneously by the example of the regular countably infinite system of points and lines pictured in Figure 1.

3. If a finite set of points, not all collinear, is such that on the circle determined by any three of the points there is always a fourth point of the set, then all the points lie on a single circle.

A set is said to be *bounded* if it lies in some circular disk of finite radius, *closed* if it includes all its *limit points*, where a point p is a limit point of a set X if every disk centered at p includes a point of X different from p . In both hypothesis and conclusion, the following theorem is closely related to Proposition **3**.

4. If a bounded closed set of points is such that the axis of symmetry for any two of its points is always an axis of symmetry for the entire set, then all the points lie on a single circle.

It is easy to see that Propositions **3** and **4** are invalid for sets that are both infinite and unbounded. Indeed, it suffices to consider the entire plane as the

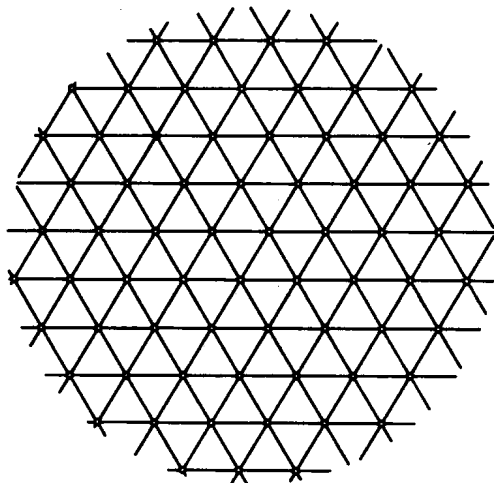


Fig. 1

set of points in question. An example consisting of countably many points may be constructed as follows: Let A_0 be a set of four points not lying on any line or circle. Then let an increasing sequence of finite sets A_n ($n = 0, 1, \dots$) be constructed recursively by setting

$$A_n = \phi(A_{n-1}) \quad (n = 1, 2, \dots),$$

where $\phi(A)$ denotes the union of all reflections of A in the various lines that are axes of symmetry for pairs of points in A . As one easily verifies, the union S of all the sets A_n is a countably infinite set with the desired property of symmetry. On the circle determined by any three points of S , there is always a fourth point of S as long as the three points do not determine an equilateral triangle (and even when they do if the construction ϕ is slightly extended).

2. Integral Distances. Commensurable Angles

We present next a group of propositions in which a role is played by the integral or rational nature of certain distances.

The set of all points, whose coordinates with respect to an orthogonal coordinate system are both integers, forms the plane *unit lattice*; its points are called *lattice points*.

5. If n lattice points form a regular n -gon ($n > 2$), then $n = 4$; thus the square is the only regular polygon that can be embedded in the unit lattice.

An ingenious proof of this was given by W. Scherrer [93]; for the case $n = 3$, see also G. Polya-G. Szegő [82], 2, page 156, problem 238.

As is seen in Figure 2, a square can be embedded in the lattice in ways other than the obvious ones.

For the angles of embedded rhombi the following is known.

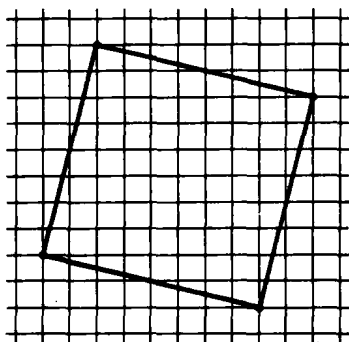


Fig. 2

6. If four lattice points determine a nonsquare rhombus with angle α , the quotient α/π is irrational; thus the square is the only rhombus that has angles commensurate with right angles and that can be embedded in the unit lattice.

Closely connected with this is an assertion about angles in Pythagorean triangles, that is, in right triangles whose sides have integral lengths.

7. If α is an acute angle of a Pythagorean triangle, the number α/π is irrational.

Propositions 6 and 7 are geometric corollaries of the following trigonometric theorem (Hadwiger [33]).

8. If $0 < \alpha < \pi/2$ and the number $\cos \alpha$ is rational, then either $\alpha = \pi/3$ or α/π is irrational.

9. If an infinite set of points is such that all of its points are at integral distances from each other, then all of the points lie on a single line.

This theorem of P. Erdős [17] may serve as prototype for a certain category of results that are especially appealing in that strong and unexpected consequences are drawn from the simplest assumptions.

Remarkably, Proposition 9 does not imply the existence of a number k_0 such that the conclusion always holds when the number k of points with exclusively integral distances is greater than k_0 . In fact, for each k there are such sets that are not linear, and even ones that have no three collinear points. Such point sets have been constructed repeatedly by M. Altwegg [1], A. Müller [77], F. Steiger [98], and others.

Following an idea of A. Müller, it is possible to describe a countably infinite point set that is dense in the unit circle, that is, intersects every arc of the circle, and has the property that all of its points are at rational distances from each other. Let P_n be the point with polar coordinates $\rho = 1$, $\theta = 2n\phi$, where ϕ is such that $\cos \phi = 4/5$, so that ϕ/π is irrational by Proposition 8.

The points of the sequence $P_n (n = 0, 1, \dots)$ are pairwise distinct, and the countably infinite set that they determine lies on the unit circle. It is dense in the circle and even uniformly distributed there, though we shall not define this notion precisely here. For the distance between any two of these points we have

$$d(P_n, P_m) = 2 |\sin (n - m)\phi|,$$

whose rationality follows from a well-known trigonometric identity and from the fact that $\sin \phi = 3/5$, $\cos \phi = 4/5$. Now, under an appropriate dilation, a similarity transformation of the plane, any k points of this set give rise to a set of k points having exclusively integral distances. Still no three of the points are collinear.

3. Hull Formation. Separability

The following group of propositions is concerned with hull formation and separation for plane point sets. First, some definitions. A set is called *convex* if with any two of its points it contains the entire line segment joining them. The *convex hull* of a set is the smallest convex set containing it. Alternatively, the convex hull is the intersection of all convex sets that contain the given set. A point is *interior* to a set if the set contains some circular disk centered at the point.

10. A point is in the convex hull of a set if, and only if, it is already in the convex hull of three or fewer points of the set.

From this it follows that the convex hull is identical with the union of all triangles whose vertices belong to the given set, or with a similar union of segments when the set is contained in a line.

11. A point is an interior point of the convex hull of a set if, and only if, it is interior to the convex hull of four or fewer points of the set.

Propositions **10** and **11** are the two-dimensional special cases of useful theorems of E. Steinitz [99] and W. Gustin [30]. See also O. Hanner-H. Rådström [41] and C. V. Robinson [90].

Two sets will be called *separable* if there is a line that does not intersect either set and that separates them from each other; the two sets then lie interior to the two half planes that are determined by the line. The following criterion for separability is due to P. Kirchberger [55] (see also H. Rademacher-I. J. Schoenberg [83]).

12. Two bounded closed sets are separable if, and only if, any two of their subsets whose union includes at most four points are separable.

13. Each set that includes at least four points can be decomposed into two nonempty disjoint subsets that are not separable.

In this connection, see F. W. Levi [69] and R. Rado [86].

4. Helly's Theorem. Transversal Problems for Ovals

We turn now to a circle of questions centered around the famous theorem of Helly. The numerous variants, Helly type theorems, which as a rule refer to ovals, form a very typical part of the combinatorial geometry of convex sets.

By an *oval* we understand here a bounded, closed convex set.

14. If each three ovals of a finite or infinite family of ovals have a common point, then all the ovals of the family have a common point.¹

This is the plane case of the well-known theorem of Helly. See E. Helly [42], J. Radon [88], D. König [64], and others. As one sees immediately from the simplest examples, the number three cannot be replaced by two. This is possible, however, with a strong assumption on the shape of the ovals. Thus the following variant is known.

15. If each two rectangles of a family of "parallel" rectangles, that is, with sides parallel to the coordinate axes, have a common point, then all the rectangles of the family have a common point.

In contrast, an oval that is not a parallelogram can be placed in three positions so that every two of the translates have a common point, but not all three. For parallelograms this is not possible. Thus parallelograms are characterized by a slightly modified version of the property expressed in Proposition 15. See B. Sz.-Nagy [78] and especially Proposition 87 below.

A corollary of Proposition 15 is Helly's theorem for the line.

16. If each two segments of a family of segments (in the line) have a common point, then all the segments of the family have a common point.

Theorems of Helly type for the circle are closely related to Proposition 16 and are useful for many applications; instead of ovals, these involve circular arcs, which obviously should lie on the same circle.

17. If a family of circular arcs, all smaller than a semicircle, is such that each three of the arcs have a common point, then all the arcs of the family have a common point.

The condition on the size of the arcs cannot be weakened, since the proposition is already false for semicircles. In fact, if four semicircles arise from two different pairs of antipodal points of the circle, then each three have a common point but not all four. Also, the number three in Proposition

¹When discussing a family of sets or points, we often have occasion to require that for each way of choosing k (not necessarily distinct) members from the family, the k members chosen have, collectively, a certain property (such as a nonempty intersection). We express this succinctly by saying that each k members have the property.

17 cannot be replaced by two; of three equal arcs that just cover the entire circle, each two have a common point but not all three. On the other hand, the following is known.

18. If a family of circular arcs, all smaller than one third of a circle, is such that each two of the arcs have a common point, then all the arcs of the family have a common point.

For the next result, we drop all assumptions on the size of the arcs.

19. If a family of circular arcs is such that each two of the arcs have a common point, then there is an antipodal pair of points such that each arc of the family includes at least one point of the pair.

In other words, there is a diameter of the circle that intersects all the arcs. Theorems of this sort were found by C. V. Robinson [90] and A. Horn-F. A. Valentine [50], among others. Some beautiful applications, such as those we shall describe later, were discovered by P. Vincensini [111].

20. If an oval can always be translated so that it is contained in the intersection of any three members of a family of ovals, then it can also be translated so that it is contained in the intersection of all the ovals of the family.

21. If an oval can always be translated so that it intersects any three members of a family of ovals, then it can also be translated so that it intersects all the members of the family.

22. If an oval can always be translated so that it contains any three members of a family of ovals, then it can also be translated so that it contains all of the members.

These are the plane cases of more general (higher-dimensional) variants of Helly's theorem, formulated by P. Vincensini [110] and V. L. Klee, Jr. [58]. For the validity of these results it is essential that the ovals should only be translated and not rotated. If the translation group is replaced by the group of rigid motions, then all three assertions become false.

We illustrate this more fully by an example connected with Proposition

21. Consider the family of n circular disks ($n > 2$) whose centers have polar coordinates $\rho = 1$ and $\theta = 2k\pi/n$ ($k = 1, \dots, n$), and whose radii are all equal to $\cos^2(\pi/n)$ or $\cos^2(\pi/n) + \cos^2(\pi/2n) - 1$ according to whether n is even or odd. As can be verified, a segment (degenerate oval) of length 2 can always be placed so as to intersect any $n - 1$ disks of the family but not to intersect all n of them. For this the segment must be suitably translated and rotated. The case $n = 8$ is illustrated by Figure 3.

23. If a family of ovals is such that each two of its members have a common point, then through each point of the plane there is a line that intersects all the ovals of the family.

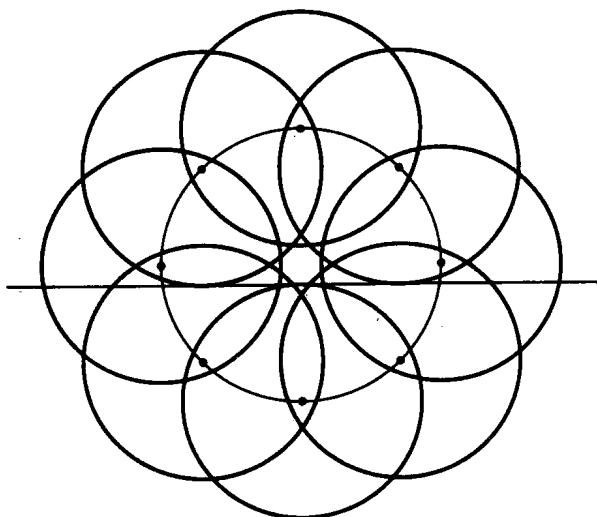


Fig. 3

24. If a family of ovals is such that each two of its members have a common point, then for each line in the plane there is a parallel line that intersects all the ovals of the family.

Propositions **23** and **24** are the plane cases of more general theorems of A. Horn [49] and V. L. Klee, Jr. [56]; they answer the question as to what can replace the conclusion of Helly's theorem when the number three is replaced by two in its hypothesis.

One may ask whether points can be replaced by lines in Helly's theorem, in the sense that an assertion of the following form is correct: If each h members of a family of ovals are intersected by a line, then there is a line that intersects all the ovals of the family. Does there exist such a Helly "stabbing number" h ?

The answer is negative! L. A. Santaló [91] has already observed that for each natural number $n > 2$ it is possible to construct a family of n ovals so that each $n - 1$ members of the family admit a common transversal, but not all n . This is illustrated also by our example in connection with Proposition **21**. Theorems of the sort mentioned can be established only under supplementary conditions on the shape and position of the ovals involved. For example, Santaló proved that all of the rectangles of a family of parallel rectangles are intersected by a line if this is the case for each six rectangles of the family. We add here the following assertion, which contains Santaló's theorem.

25. If each three rectangles of a family of parallel rectangles are intersected by an ascending line, then there is an ascending line that intersects all the rectangles of the family.