# RELATIVISTIC QUANTUM THEORY

Volume 4 of Course of Theoretical Physics

## Part 2

**E. M. Lifshitz**
**L. P. Pitaevskiĭ**

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# PREFACE

THIS is a direct continuation of Part 1, which we wrote jointly with V. B. Berestetskiĭ, and the numbering of the chapters and sections in the two parts forms one series.

The first three chapters deal with the theory of radiative corrections, and thus complete the exposition of quantum electrodynamics. Here again we have tried not only to emphasise the physical principles of the methods used, but also to display important points of technique, and the actual calculations are therefore given in some detail.

It was a part of L. D. Landau's original concept of the *Course of Theoretical Physics* that it should describe only those results which can be reasonably regarded as established and embodied in a definite theoretical system. The treatment should not approach too closely the "frontiers" of theoretical physics. It is, of course, especially difficult to apply this criterion at the present time to a subject such as the theory of weak and strong interactions. Moreover, an account of various isolated topics would be of little use to those who are particularly concerned with this subject, and would be of no interest to those who are not. For these reasons, we have decided to limit ourselves in the present book to just the dynamical symmetry of strong interactions and the "phenomenological" theory of weak interactions. We hope that, despite this limitation, the book will be useful to a fairly large number of readers.

As in the other volumes of the *Course*, the references to the literature that are given here make no claim to completeness or to the indication of priority. Their sole purpose is as a guide to material lying somewhat outside the scope of our account, but closely related to it. Even so, the nature of the subject is such that the choice of references is difficult and uncertain.

References to earlier volumes in the course are made as shown on page xv of Part 1.

We should like to thank B. L. Ioffe and I. S. Shapiro, who read through the book and made a number of useful comments. We are grateful also to Ya. I. Azimov and A. A. Varfolomeev for advice, and we have made profitable use of the book by L. B. Okun'† in dealing with the theory of weak interactions. We are deeply indebted to Dr. Sykes and Dr. Bell for their accurate translation and also for their comments that have enabled us to eliminate some imprecisions of presentation.

E. M. LIFSHITZ
L. P. PITAEVSKIĬ

† *Weak Interaction of Elementary Particles*, Pergamon Press, Oxford, 1965.

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CHAPTER XI

# EXACT PROPAGATORS AND VERTEX PARTS

## §99. Field operators in the Heisenberg representation

HITHERTO, in considering various specific processes in electrodynamics, we have used only the first non-vanishing approximation of perturbation theory. We shall now go on to discuss the effects which occur in higher approximations. These are called *radiative corrections*.

A better understanding of the structure of the higher approximations can be obtained by first examining some general properties of exact scattering amplitudes (i.e. those which have not been expanded in powers of $e^2$). We have seen in §73 that the successive terms of the series in perturbation theory can be expressed in terms of the field operators in the interaction representation, whose time dependence is determined by the Hamiltonian $\mathrm{H}_0$ of a system of free particles. The exact scattering amplitudes, however, are more conveniently expressed in terms of the field operators in the Heisenberg representation, where the time dependence is determined by the exact Hamiltonian $\mathrm{H} = \mathrm{H}_0 + \mathrm{V}$ of a system of interacting particles.

The general rule for constructing the Heisenberg operators gives

$$\psi(x) \equiv \psi(t, \boldsymbol{r}) = e^{i\mathrm{H}t}\psi(\boldsymbol{r})e^{-i\mathrm{H}t}, \tag{99.1}$$

and similarly for $\bar{\psi}(x)$ and $\mathrm{A}(x)$, $\psi(\boldsymbol{r})$, etc., being time-independent (Schrödinger) operators.† It may be noted immediately that the Heisenberg operators for a given time obey the same commutation rules as the operators in the Schrödinger representation or the interaction representation: for example,

$$\{\psi_i(t, \boldsymbol{r})\bar{\psi}_k(t, \boldsymbol{r}')\}_+ = e^{i\mathrm{H}t}\{\psi_i(\boldsymbol{r})\bar{\psi}_k(\boldsymbol{r}')\}_+ e^{-i\mathrm{H}t} = \gamma^0_{ik}\,\delta(\boldsymbol{r}-\boldsymbol{r}'); \tag{99.2}$$

cf. (76.6). Similarly, the operators $\psi(t, \boldsymbol{r})$ and $\mathrm{A}(t, \boldsymbol{r}')$ commute:

$$\{\psi_i(t, \boldsymbol{r})\,\mathrm{A}(t, \boldsymbol{r}')\}_- = 0,$$

but this does *not* hold good for operators pertaining to different times.

† In this chapter, operators with a time argument belong to the Heisenberg representation; those in the interaction representation will be given the suffix int.

The "equation of motion" satisfied by the Heisenberg $\psi$-operator can be derived from the general formula *QM* (13.7):

$$-i\frac{\partial\psi(x)}{\partial t} = \mathrm{H}\psi(x)-\psi(x)\,\mathrm{H}. \tag{99.3}$$

The Schrödinger and Heisenberg representations are the same as regards the Hamiltonian, which is expressed in the same way in terms of the field operators. Here, to calculate the right-hand side of (99.3), we may omit from the Hamiltonian the part which depends only on the operator $\mathrm{A}(x)$ (the Hamiltonian of the free electromagnetic field), since this part commutes with $\psi(x)$. According to (21.13) and (43.3),

$$\begin{aligned}\mathrm{H} &= \int \psi^*(t,\boldsymbol{r})\,(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\,\psi(t,\boldsymbol{r})\,d^3x+e\int\bar{\psi}(t,\boldsymbol{r})\,\hat{\mathrm{A}}(t,\boldsymbol{r})\,\psi(t,\boldsymbol{r})\,d^3x\\ &= \int\bar{\psi}(t,\boldsymbol{r})\left\{\gamma\mathrm{p}+m+e\hat{\mathrm{A}}(t,\boldsymbol{r})\right\}\psi(t,\boldsymbol{r})\,d^3x.\end{aligned} \tag{99.4}$$

When the commutator $\{\mathrm{H}\psi(t,\boldsymbol{r})\}_-$ is calculated from (99.2) and the delta function is eliminated by integration over $d^3x$, we get

$$\left(\hat{\mathrm{p}}-e\hat{\mathrm{A}}-m\right)\psi(t,\boldsymbol{r}) = 0. \tag{99.5}$$

As we should expect, the operator $\psi(t,\boldsymbol{r})$ satisfies an equation which is formally the same as Dirac's equation.

The equation for the electromagnetic field operator $\mathrm{A}(t,\boldsymbol{r})$ is obvious from the correlation with the classical case. When that case applies, i.e. when the occupation numbers are large (cf. §5), the operator equation must become the classical Maxwell's equation for the potentials, *Fields* (30.2), after averaging over the state of the field. It is therefore clear that the equation for the operator is simply the same as Maxwell's equation, so that we have (for an arbitrary gauge)

$$\partial^\nu\partial_\mu\mathrm{A}^\mu(x)-\partial^\mu\partial_\mu\mathrm{A}^\nu(x) = -4\pi e\mathrm{j}^\nu(x), \tag{99.6}$$

where $\mathrm{j}^\nu(x) = \bar{\psi}(x)\,\gamma^\nu\,\psi(x)$ is the current operator, satisfying identically the equation of continuity†

$$\partial_\nu\mathrm{j}^\nu(x) = 0. \tag{99.7}$$

It is important to note that the equations (99.6) are linear in $\mathrm{A}^\mu$ and $\mathrm{j}^\mu$, and the question of the sequence of these operators does not arise.

Like the similar equations for wave functions, the operator equations (99.6) and (99.7) are invariant under the gauge transformation

$$\left.\begin{aligned}\mathrm{A}_\mu(x) &\to \mathrm{A}_\mu(x)-\partial_\mu\chi(x),\\ \psi(x) &\to \psi(x)e^{ie\chi},\\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{-ie\chi},\end{aligned}\right\} \tag{99.8}$$

where $\chi(x)$ is any real operator which commutes (at a particular time) with $\psi$.‡

† The operators $\mathrm{A}^\mu_{\mathrm{int}}(x)$ corresponding to the free electromagnetic field satisfy the same equation with zero on the right-hand side:

$$\partial^\nu\partial_\mu\mathrm{A}^\mu_{\mathrm{int}}(x)-\partial^\mu\partial_\mu\mathrm{A}^\nu_{\mathrm{int}}(x) = 0. \tag{99.6a}$$

‡ This refers specifically to the Heisenberg $\psi$-operators. In the interaction representation, the gauge transformation of the electromagnetic potentials does not affect the $\psi$-operators.

Let us now ascertain the relationship between the operators in the Heisenberg representation and those in the interaction representation. To simplify the discussion, it is convenient to make the formal assumption (which will not affect the final result) that the interaction $V(t)$ is adiabatically "switched on" from $t = -\infty$ to finite times. Then the Heisenberg and interaction representations are the same for $t \to -\infty$, and the wave functions of the system, $\Phi$ and $\Phi_{\text{int}}$, are the same:

$$\Phi_{\text{int}}(t = -\infty) = \Phi. \tag{99.9}$$

But the wave function in the Heisenberg representation is independent of time (since the whole of the time dependence is in the operators); in the interaction representation, the time dependence of the wave function is given by (73.7):

$$\Phi_{\text{int}}(t) = S(t, -\infty)\,\Phi_{\text{int}}(-\infty), \tag{99.10}$$

where†

$$S(t_2, t_1) = T \exp\left\{-i \int_{t_1}^{t_2} V(t')\,dt'\right\}. \tag{99.11}$$

Comparison of (99.10) and (99.9) gives

$$\Phi_{\text{int}}(t) = S(t, -\infty)\Phi \tag{99.12}$$

as the relationship between the wave functions in the two representations. The operator transformation formula is similarly

$$\begin{aligned} \psi(t, \boldsymbol{r}) &= S^{-1}(t, -\infty)\ \psi_{\text{int}}(t, \boldsymbol{r})\, S(t, -\infty) \\ &= S(-\infty, t)\ \psi_{\text{int}}(t, \boldsymbol{r})\, S(t, -\infty), \end{aligned} \tag{99.13}$$

and likewise for $\bar{\psi}$ and A.

One further general remark may be added. It has already been mentioned more than once that, in relativistic quantum theory, the physical significance of the field operators is very limited because the zero-point fluctuations are infinite. This is even more true of operators in the Heisenberg representation, which contain also divergences due to the interaction. In this chapter, §§99–106 deal with the formal theory, which ignores the question of eliminating these singularities and which treats all quantities as if they were finite. The results thus obtained have mainly heuristic value: they lead to a fuller understanding of the significance of the expansions given by perturbation theory, and they may also remain valid in some form in a future theory which is free from the present difficulties.‡

† The following properties of S are obvious:

$$\begin{aligned} S(t, t_1)\, S(t_1, t_0) &= S(t, t_0), \\ S(t, t_0)\, S(t_0, t) &= 1. \end{aligned} \tag{99.11a}$$

‡ The mathematical formalism developed here can be used also in quantum statistical physics, where the divergences typical of field theory do not occur.

## §100. The exact photon propagator

The concepts of exact propagators play a central role in the formalism of the exact theory (i.e. without expansion in powers of $e^2$).†

The *exact photon propagator* (denoted by the script letter $\mathcal{D}$) is defined by

$$\mathcal{D}_{\mu\nu}(x-x') = i\langle 0\,|\,\mathrm{T}\mathrm{A}_\mu(x)\,\mathrm{A}_\nu(x')\,|\,0\rangle, \tag{100.1}$$

where $\mathrm{A}_\mu(x)$ are Heisenberg operators, in contrast to the definition (77.1):

$$\mathrm{D}_{\mu\nu}(x-x') = i\langle 0\,|\,\mathrm{T}\mathrm{A}^{\mathrm{int}}_\mu(x)\,\mathrm{A}^{\mathrm{int}}_\nu(x')\,|\,0\rangle, \tag{100.2}$$

in which the operators in the interaction representation were used. The function (100.2) may be called the *free* (or *bare*)*-photon propagator* to distinguish it from the exact propagator (100.1).

Since the mean value in (100.1) cannot be exactly calculated, it is impossible to obtain an exact analytical expression for $\mathcal{D}_{\mu\nu}$, although the definition does lead to some general properties of this function, as will be discussed in §108; here we shall consider the calculation of $\mathcal{D}_{\mu\nu}$ by perturbation theory, using the diagram technique. For this purpose, we must express $\mathcal{D}_{\mu\nu}$ in terms of the operators in the interaction representation.

First, let $t > t'$. Using the relationship between $\mathrm{A}(x)$ and $\mathrm{A}_{\mathrm{int}}(x)$ (cf. (99.13)), we can write

$$\begin{aligned}\mathcal{D}_{\mu\nu}(x-x') &= i\langle 0\,|\,\mathrm{A}_\mu(x)\,\mathrm{A}_\nu(x')\,|\,0\rangle \\ &= i\langle 0\,|\,\mathrm{S}(-\infty, t)\,\mathrm{A}^{\mathrm{int}}_\mu(x)\,\mathrm{S}(t, -\infty)\,\mathrm{S}(-\infty, t')\times \\ &\quad \times \mathrm{A}^{\mathrm{int}}_\nu(x')\,\mathrm{S}(t', -\infty)\,|\,0\rangle.\end{aligned}$$

According to (99.11a) we can make the substitutions

$$\begin{aligned}\mathrm{S}(t, -\infty)\,\mathrm{S}(-\infty, t') &= \mathrm{S}(t, t'), \\ \mathrm{S}(-\infty, t) &= \mathrm{S}(-\infty, +\infty)\,\mathrm{S}(\infty, t).\end{aligned}$$

Then

$$\mathcal{D}_{\mu\nu}(x-x') = i\langle 0\,|\,\mathrm{S}^{-1}[\mathrm{S}(\infty, t)\,\mathrm{A}^{\mathrm{int}}_\mu(x)\,\mathrm{S}(t, t')\,\mathrm{A}^{\mathrm{int}}_\nu(x')\mathrm{S}(t', -\infty)]\,|\,0\rangle, \tag{100.3}$$

with

$$\mathrm{S} \equiv \mathrm{S}(+\infty, -\infty). \tag{100.4}$$

Since, according to the definition (99.11), $\mathrm{S}(t_2, t_1)$ includes only operators for times between $t_1$ and $t_2$ arranged in chronological sequence, it is evident that all the operator factors in the brackets in (100.3) are in order of decreasing time from left to right. If the time-ordering

† These concepts were introduced by F. J. Dyson (1949), who also developed essentially the whole of the treatment given in this chapter.

symbol T is placed before the bracket, we can rearrange the factors in any manner, since the operator T will automatically put them in the necessary order. Then we write the bracket as

$$[\ldots] = \mathrm{T}[\mathrm{A}_\mu^{\mathrm{int}}(x)\,\mathrm{A}_\nu^{\mathrm{int}}(x')\,\mathrm{S}(\infty, t)\,\mathrm{S}(t, t')\,\mathrm{S}(t', -\infty)] = \mathrm{T}[\mathrm{A}_\mu^{\mathrm{int}}(x)\,\mathrm{A}_\nu^{\mathrm{int}}(x')\,\mathrm{S}].$$

Thus

$$\mathcal{D}_{\mu\nu}(x-x') = i\langle 0\,|\,\mathrm{S}^{-1}\mathrm{T}[\mathrm{A}_\mu^{\mathrm{int}}(x)\,\mathrm{A}_\nu^{\mathrm{int}}(x')\,\mathrm{S}]\,|\,0\rangle. \tag{100.5}$$

It is easily shown by a similar argument that this formula is also valid if $t < t'$.

We shall now prove that the factor $\mathrm{S}^{-1}$ can be taken outside the averaging over the vacuum to form a phase factor. To do so, we recall that the Heisenberg vacuum wave function $\Phi$ is the same as the value $\Phi_{\mathrm{int}}(-\infty)$ of the wave function of the same state in the interaction representation (see (99.9)). From (73.8),

$$\mathrm{S}\Phi_{\mathrm{int}}(-\infty) \equiv \mathrm{S}(+\infty, -\infty)\,\Phi_{\mathrm{int}}(-\infty) = \Phi_{\mathrm{int}}(+\infty).$$

The vacuum is a strictly stationary state, in which no spontaneous processes of particle generation can occur. In other words, in the course of time the vacuum remains the vacuum; this means that $\Phi_{\mathrm{int}}(+\infty)$ can differ from $\Phi_{\mathrm{int}}(-\infty)$ only by a phase factor $e^{i\alpha}$. Hence

$$\mathrm{S}\Phi_{\mathrm{int}}(-\infty) = e^{i\alpha}\,\Phi_{\mathrm{int}}(-\infty) = \langle 0\,|\,\mathrm{S}\,|\,0\rangle\,\Phi_{\mathrm{int}}(-\infty), \tag{100.6}$$

or, taking the complex conjugate and using the unitarity of the operator S,

$$\Phi_{\mathrm{int}}^*(-\infty)\mathrm{S}^{-1} = \langle 0\,|\,\mathrm{S}\,|\,0\rangle^{-1}\,\Phi_{\mathrm{int}}^*(-\infty).$$

Hence it is clear that (100.5) can be written

$$\mathcal{D}_{\mu\nu}(x-x') = i\,\frac{\langle 0\,|\,\mathrm{T}\mathrm{A}_\mu^{\mathrm{int}}(x)\,\mathrm{A}_\nu^{\mathrm{int}}(x')\,\mathrm{S}\,|\,0\rangle}{\langle 0\,|\,\mathrm{S}\,|\,0\rangle}. \tag{100.7}$$

Substituting in the numerator and the denominator the expansion (73.10) for S and averaging by means of Wick's theorem (§78), we get an expansion of $\mathcal{D}_{\mu\nu}$ in powers of $e^2$.

In the numerator of (100.7), the quantities to be averaged differ from the matrix elements of the type (78.1) only in that the "external" photon creation and annihilation operators are replaced by $\mathrm{A}_\mu^{\mathrm{int}}(x)$ and $\mathrm{A}_\nu^{\mathrm{int}}(x')$. Since all the factors in the products to be averaged are preceded by the time-ordering symbol, the pairwise contractions of these operators with the "internal" operators $\mathrm{A}^{\mathrm{int}}(x_1)$, $\mathrm{A}^{\mathrm{int}}(x_2)$, ... will give the photon propagators $D_{\mu\nu}$. Thus the results of the averaging are expressed by sets of diagrams with two free ends, constructed in accordance with the rules in §78, except that propagators $D_{\mu\nu}$, not the amplitudes $e$ of real photons, correspond to external (and internal) photon lines. In the zero-order approximation, with $\mathrm{S} = 1$, the numerator of (100.7) is simply $D_{\mu\nu}(x-x')$. The next non-zero terms will be proportional to $e^2$. They are represented by a set of diagrams having two free ends

and two vertices:


(100.8)

The second of these diagrams consists of two disconnected parts: a broken line (corresponding to $-iD_{\mu\nu}$) and a closed loop. The separation of the parts of the diagram signifies that the corresponding analytical expression separates into two independent factors. On adding to the diagrams (100.8) the zero-order approximation diagram (a single broken line) and "taking it outside the brackets", we find that the numerator in (100.7) is, as far as second-order terms,


The expression $\langle 0 \mid S \mid 0 \rangle$ in the denominator of (100.7) is the amplitude of the "transition" from the vacuum to the vacuum. Its expansion therefore contains only diagrams without free ends. In the zero-order approximation, $\langle 0 \mid S \mid 0 \rangle = 1$, and as far as second-order terms we have


When the numerator is divided by the denominator we get, to the same order, the expression


Thus the diagram with the detached loop does not occur in the result. This is a general theorem. Having regard to the way in which the diagrams are constructed which correspond to the numerator and denominator in (100.7), we can easily see that the role of the denominator $\langle 0 \mid S \mid 0 \rangle$ is simply to ensure that in all orders of perturbation theory the exact propagator $\mathcal{D}_{\mu\nu}$ will be represented only by diagrams which do not contain separated parts.

The diagrams with no free ends, forming closed loops, have no physical significance and need not be taken into account, quite apart from the fact that they disappear when the propagator $\mathcal{D}$ is formed. Such loops represent radiative corrections to the diagonal element of the $S$-matrix for a vacuum–vacuum transition; but, according to (100.6), the sum of all these loops, together with the unity given by the zero-order approximation, gives only an unimportant phase factor, which cannot affect any physical results.

The change from the coordinate representation to the momentum representation is made in the usual way. For example, in the second-order approximation of perturbation theory,

the propagator $-i\mathcal{D}_{\mu\nu}(k)$, which will be shown by a thick broken line, is the sum

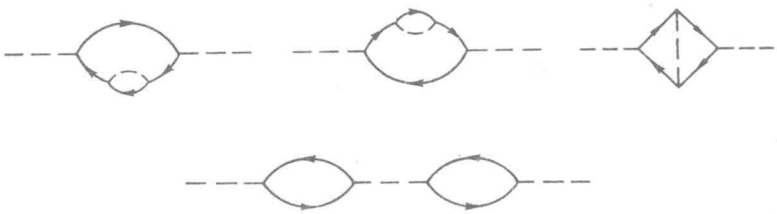

(100.9)

in which all the diagrams are calculated by the general rules given in §78 except that factors $-iD_{\mu\nu}(k)$ are assigned to the external as well as the internal photon lines. In analytical form, we therefore have†

$$\mathcal{D}_{\mu\nu}(k) \approx D_{\mu\nu}(k) + ie^2 D_{\mu\lambda}(k) \int \mathrm{tr}\, \gamma^\lambda G(p+k)\, \gamma^\varrho G(p)\, \frac{d^4p}{(2\pi)^4} D_{\varrho\nu}(k); \qquad (100.10)$$

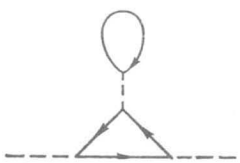
the bispinor indices of the matrices $\gamma$ and $G$ are, as usual, omitted.

The terms in subsequent approximations are constructed in a similar manner, and are represented by sets of diagrams having two external photon lines and the appropriate number of vertices. For example, the terms in $e^4$ correspond to the following four-vertex diagrams:



(100.11)

The diagram



also has four vertices; its upper part is a loop formed by a single "self-closed" electron line.‡ Such a loop corresponds to the contraction $\bar{\psi}(x)\psi(x)$, i.e. to the value of the current averaged over the vacuum: $\langle 0 \mid \mathrm{j}(x) \mid 0 \rangle$. But, by the definition of the vacuum, this quantity must be zero identically, and the identity cannot of course be altered by any further radiative corrections to such a loop.§ Thus no diagrams having "self-closed" electron lines need be considered in any approximation.

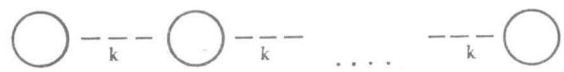
The part of a diagram which lies between two (external or internal) photon lines is called a *photon self-energy part*. In the general case, it can itself be divided into parts joined in pairs

† The factor $-1$ from the closed electron loop must be taken into account when deriving the signs.

‡ Unlike the loop in (100.8b), which is formed by two separate electron lines.

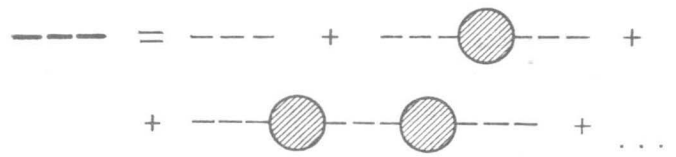
§ Although a direct calculation from the diagrams would lead to divergent integrals.

by a single photon line, i.e. it has a structure of the form




where the circles denote parts which cannot be further subdivided in the same manner; such parts are said to be *compact* or *proper*. For example, the first three of the four fourth-order self-energy parts (100.11) are compact.

Let $i\mathcal{P}_{\mu\nu}/4\pi$ denote the sum of the infinity of compact self-energy parts. The function $\mathcal{P}_{\mu\nu}(k)$ is called the *polarisation operator*. When the diagrams are classified by the number of compact parts which they contain, the exact propagator $\mathcal{D}_{\mu\nu}$ can be put in the form of a series



where $i\mathcal{P}_{\mu\nu}/4\pi$ corresponds to each shaded circle. The analytical form of this series is

$$\begin{aligned}\mathcal{D} &= D+D\frac{\mathcal{P}}{4\pi}D+D\frac{\mathcal{P}}{4\pi}D\frac{\mathcal{P}}{4\pi}D+\ \ldots\\ &= D\left\{1+\frac{\mathcal{P}}{4\pi}\left[D+D\frac{\mathcal{P}}{4\pi}D+\ \ldots\right]\right\},\end{aligned}\tag{100.12}$$

where the indices are omitted, for brevity. The series in the brackets is again $\mathcal{D}$. Hence

$$\mathcal{D}_{\mu\nu}(k) = D_{\mu\nu}(k)+D_{\mu\lambda}(k)\frac{\mathcal{P}^{\lambda\varrho}(k)}{4\pi}\mathcal{D}_{\varrho\nu}(k).\tag{100.13}$$

Multiplying this equation on the left by the inverse tensor $(D^{-1})^{\tau\mu}$ and on the right by $(\mathcal{D}^{-1})^{\nu\sigma}$, and renaming the indices, we get the equivalent form

$$\mathcal{D}^{-1}{}_{\mu\nu} = D^{-1}{}_{\mu\nu}-\frac{1}{4\pi}\mathcal{P}_{\mu\nu}.\tag{100.14}$$

It must be emphasised that writing $\mathcal{D}$ in the form (100.12) assumes that the diagrams can be broken down into simpler parts calculated by the general rules of the diagram technique, and that the combination of such parts gives the correct expressions for the entire diagrams. The admissibility of this breakdown of the diagrams is an important and by no means trivial feature of the diagram technique, which arises from the fact that the overall numerical factor in the diagram does not depend on the order of the diagram.

The same property enables us to use the function $\mathcal{D}$ (assumed known) to simplify the calculations of the radiative corrections to the amplitudes of various scattering processes: instead of treating afresh each time the diagrams with different corrections to the internal

photon lines, we can simply make these lines thick, i.e. assign to them the propagators $\mathcal{D}$ (instead of $D$) in the appropriate approximation.

If the photon line corresponds to a real and not a virtual photon, i.e. if it is a free end of the whole diagram, the application to it of all the self-energy corrections gives what is called an *effective external line*. It corresponds to the expression obtained from (100.13) by replacing the factor $D$ by the polarisation amplitude of the real photon:

$$e_\mu + \mathcal{D}_{\mu\varrho}(k)\,\frac{\mathcal{P}^{\varrho\lambda}(k)}{4\pi}\,e_\lambda . \tag{100.15}$$

For an external-field line, $e_\mu$ in this expression is to be replaced by $A^{(e)}_\mu$.

The discussion in §77 of the tensor structure and the gauge non-uniqueness of the approximate propagator $D_{\mu\nu}$ applies to the exact function $\mathcal{D}_{\mu\nu}$ also. Considering only the relativistically invariant representations of this function, we can write it in the general form

$$\mathcal{D}_{\mu\nu}(k) = \mathcal{D}(k^2)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\mathcal{D}^{(l)}(k^2)\,\frac{k_\mu k_\nu}{k^2}\,; \tag{100.16}$$

the first term corresponds to the Landau gauge, and in the second term $\mathcal{D}^{(l)}$ is a gauge-arbitrary function. The corresponding form of the approximate propagator† is

$$D_{\mu\nu}(k) = D(k^2)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+D^{(l)}(k^2)\,\frac{k_\mu k_\nu}{k^2}\,. \tag{100.17}$$

The longitudinal part $\mathcal{D}^{(l)}$ of the propagator is related to the longitudinal part of the potential vector, which has no physical significance. It is therefore not concerned in the interaction and is unaffected by the latter, so that

$$\mathcal{D}^{(l)} = D^{(l)}.$$

The inverse tensors must, by definition, satisfy the equations

$$\mathcal{D}^{-1}{}_{\mu\nu}\mathcal{D}^{\lambda\nu} = \delta^\lambda_\mu\,, \qquad D^{-1}{}_{\mu\nu}D^{\lambda\nu} = \delta^\lambda_\mu\,.$$

When the original tensors have the form (100.16) or (100.17), the inverse tensors are

$$\begin{aligned}
\mathcal{D}^{-1}{}_{\mu\nu} &= \frac{1}{\mathcal{D}}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\frac{1}{D^{(l)}}\,\frac{k_\mu k_\nu}{k^2}\,,\\
D^{-1}{}_{\mu\nu} &= \frac{1}{D}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\frac{1}{D^{(l)}}\,\frac{k_\mu k_\nu}{k^2}\,.
\end{aligned} \tag{100.18}$$

From (100.14), the polarisation operator $\mathcal{P}_{\mu\nu}$ is a transverse tensor:

$$\mathcal{P}_{\mu\nu} = \mathcal{P}(k^2)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right), \tag{100.19}$$

† In this formula $D^{(l)}$ is not the same as in (77.3).

where $\mathcal{P} = k^2 - 4\pi/\mathcal{D}$, or

$$\mathcal{D}(k^2) = \frac{4\pi}{k^2[1-\mathcal{P}(k^2)/k^2]}\,. \tag{100.20}$$

Thus the polarisation operator, unlike the photon propagator itself, is a gauge-invariant quantity.

## §101. The self-energy function of the photon

In order to examine further the analytical properties of the photon propagator, it is useful to define, as well as the polarisation operator, another auxiliary function $\Pi_{\mu\nu}(k)$, called the *self-energy function of the photon*: $i\Pi_{\mu\nu}/4\pi$ is defined as the sum of all self-energy photon parts (not only the compact ones). If this sum is represented in the diagram by a square, we can write the exact propagator as the sum


i.e.

$$\mathcal{D}_{\mu\nu} = D_{\mu\nu} + D_{\mu\lambda}\frac{\Pi^{\lambda\varrho}}{4\pi}D_{\varrho\nu}\,. \tag{101.1}$$

Hence, expressing $\Pi_{\mu\nu}$ as

$$\frac{1}{4\pi}\Pi_{\mu\nu} = D^{-1}{}_{\mu\lambda}\,\mathcal{D}^{\lambda\varrho}D^{-1}{}_{\varrho\nu} - D^{-1}{}_{\mu\nu}$$

and substituting (100.16)–(100.18) followed by (100.20), we get

$$\Pi_{\mu\nu} = \Pi(k^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \quad \Pi = \frac{\mathcal{P}}{1-\mathcal{P}/k^2}\,. \tag{101.2}$$

Thus $\Pi_{\mu\nu}$, like $\mathcal{P}_{\mu\nu}$, is a gauge-invariant tensor.

The usefulness of $\Pi_{\mu\nu}$ arises from the expression for it in the coordinate representation. This is easily found by noting that the equation

$$\frac{1}{4\pi}\Pi_{\mu\nu}(k) = D^{-1}{}_{\mu\lambda}\,D^{-1}{}_{\varrho\nu}\{\mathcal{D}^{\lambda\varrho}(k) - D^{\lambda\varrho}(k)\},$$

in which

$$D^{-1}{}_{\mu\lambda}(k) = \frac{1}{4\pi}(k^2 g_{\mu\nu} - k_\mu k_\nu) + \frac{1}{D^{(l)}}\frac{k_\mu k_\nu}{k^2}\,,$$

can be written in the coordinate representation

$$\Pi_{\mu\nu}(x-x') = \frac{1}{4\pi}(\partial_\mu\partial_\lambda - g_{\mu\lambda}\,\partial_\sigma\partial^\sigma)(\partial'_\nu\partial'_\varrho - g_{\nu\varrho}\,\partial'_\sigma\partial'^\sigma)\{\mathcal{D}^{\lambda\varrho}(x-x') - D^{\lambda\varrho}(x-x')\},$$

since the tensor $\mathcal{D}^{\lambda\varrho} - D^{\lambda\varrho}$ is transverse. In order to carry out the differentiation, we must substitute

$$\mathcal{D}^{\lambda\varrho}(x-x') - D^{\lambda\varrho}(x-x') = i\langle 0 \,|\, \mathrm{T}\mathrm{A}^{\lambda}(x)\,\mathrm{A}^{\varrho}(x') - \mathrm{T}\mathrm{A}^{\lambda}_{\mathrm{int}}(x)\,\mathrm{A}^{\varrho}_{\mathrm{int}}(x') \,|\, 0\rangle. \tag{101.3}$$

In §76 we have seen that the differentiation of a T product generally demands caution, because the product has discontinuities. But the difference that is to be averaged in (101.3) is continuous, and so are its first derivatives, since the commutation rules are the same for the components of the operators $\mathrm{A}^{\lambda}(x)$ and $\mathrm{A}^{\lambda}_{\mathrm{int}}(x)$ for a given time, and the corresponding discontinuities cancel out (cf. §76). The difference in (101.3) may therefore be differentiated under the symbol T. According to (99.6) and (99.6a), the result is the required expression

$$\Pi_{\mu\nu}(x-x') = 4\pi i e^2 \langle 0 \,|\, \mathrm{T}\mathrm{j}_{\mu}(x)\,\mathrm{j}_{\nu}(x') \,|\, 0\rangle. \tag{101.4}$$

This shows explicitly the gauge-invariance of $\Pi_{\mu\nu}$, since the current operators are gauge-invariant.

From (101.4) we can derive an important integral form of this function. According to (101.2), it is sufficient to consider the scalar function $\Pi = \frac{1}{3}\Pi^{\mu}_{\mu}$. In the coordinate representation,

$$\begin{aligned} \Pi(x-x') &= \frac{4\pi}{3} ie^2 \langle 0 \,|\, \mathrm{T}\mathrm{j}_{\mu}(x)\,\mathrm{j}^{\mu}(x') \,|\, 0\rangle \\ &= \frac{4\pi}{3} ie^2 \begin{cases} \sum\limits_n \langle 0 \,|\, \mathrm{j}_{\mu}(x) \,|\, n\rangle \langle n \,|\, \mathrm{j}^{\mu}(x') \,|\, 0\rangle & \text{for} \quad t > t', \\ \sum\limits_n \langle 0 \,|\, \mathrm{j}_{\mu}(x') \,|\, n\rangle \langle n \,|\, \mathrm{j}^{\mu}(x) \,|\, 0\rangle & \text{for} \quad t < t', \end{cases} \end{aligned} \tag{101.5}$$

where $n$ labels the states of the system electromagnetic field + electron–positron field.† Since the current operator $\mathrm{j}(x)$ depends on $x^{\mu} = (t, \boldsymbol{r})$, its matrix elements also depend on $x$. The relationship can be found explicitly by taking as the states $|\, n\rangle$ states which have definite values of the total 4-momentum.

The time dependence of the current matrix elements, like that of any Heisenberg operator, is given by

$$\langle n \,|\, \mathrm{j}^{\mu}(t, \boldsymbol{r}) \,|\, m\rangle = \langle n \,|\, \mathrm{j}^{\mu}(\boldsymbol{r}) \,|\, m\rangle e^{-i(E_m - E_n)t},$$

where $E_n$ and $E_m$ are the energies of the states $|\, n\rangle$ and $|\, m\rangle$, and $\mathrm{j}(\boldsymbol{r})$ is the Schrödinger operator.

To determine the coordinate dependence of the matrix elements, we consider the operator $\mathrm{j}(\boldsymbol{r})$ as being the result of transforming the operator $\mathrm{j}(0)$ by a parallel translation over the distance $\boldsymbol{r}$. The operator of this translation is $\exp(i\boldsymbol{r}\cdot\mathbf{P})$, where $\mathbf{P}$ is the total momentum operator of the system (see *QM*, §15, Problem 1). Using the general rule for the

† The current operator conserves charge; hence the states $|\, n\rangle$, combining with the vacuum $|\, 0\rangle$, can contain only the same numbers of electrons and positrons.