Foundations of Engineering Mechanics

Yury Vetyukov

Nonlinear Mechanics of Thin-Walled Structures

Asymptotics, Direct Approach and Numerical Analysis

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Preface

The recent progress in the theoretical mechanics of solids is often not regarded by the engineering community as a basis for the analysis of practical problems. Despite the high level of modern theories of thin-walled continua, the vast majority of numerical methods and solutions rest upon approximations of three-dimensional fields over the thickness. But "there is nothing more practical than a good theory" (attributed to L. Boltzmann), and the intent of this book is to bridge the gap between the theoreticians and the structural engineers. Modern methods of analysis contribute to the elegance and efficiency of the developed theoretical formulations, and finally, to the trustworthiness of numerical schemes. Their simplicity is demonstrated in the book by the source code for modeling complicated behavior of thin-walled structures, which is possible with modern high-level simulation environments such as Wolfram's *Mathematica*.

The science of mechanics resides at the border between physics and mathematics. It has its own mentality and operates with its own criteria. The appropriate level of mathematical strictness is well demonstrated by the known joke:
A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train. "Aha," says the engineer, "I see that Scottish sheep are black." "Hmm," says the physicist, "You mean that some Scottish sheep are black." "No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"
Mathematics provides us with a handy toolbox for breaking new grounds, but experience shows that a physical way of thinking is often required for pioneering work in mechanics.

The scope of this book includes mechanical models of classical rods, plates, shells, and thin-walled rods of open profile, which are unified by the use of common methods of research. Classical theories of thin structures arise when the two ways of analysis meet and mutually complement each other. The procedure of asymptotic splitting in the three-dimensional model of the structure and the direct approach to an idealized dimensionally reduced continuum with the methods of Lagrangian mechanics constitute a very concise and formal method to developing geometrically nonlinear theories with a high level of consistency. These analytical technologies play a central role in the theoretical parts of the book, which is counterbalanced by an extensive demonstration of possibilities of numerical analysis with the developed models. The presented material is self-sufficient, and the basic notions are discussed in the introductory part. Nevertheless, preliminary knowledge in the theory of elasticity, analytical mechanics and basic ideas of the method of finite elements should be recommended. Many theoretical and especially numerical aspects are illustrated by examples of mathematical modeling, performed with the *Mathematica* software. This modern language of science allows complicated simulations to be performed residing at the problem-oriented level without the need of programming sophisticated algorithms of numerical mathematics. A short reference for *Mathematica* is provided in Chap. 6. The source code of the simulations is an important constituent of the text of the book. It practically illustrates the proposed methods of modeling and provides the simulation results in their "naked" form, as nothing is hidden and everything can be reproduced. The files with these simulations are available for download at the SpringerLink online platform,[1] which grants the reader a possibility to experiment with the developed algorithms or to enhance them, avoiding the burden of retyping the necessary source code.

The author's understanding and aesthetic feeling of mechanics were greatly influenced by the learning and work together with Prof. Vladimir Eliseev (Yeliseyev), who is still carrying the spirit of the school of mechanics, founded at the Polytechnic University of St. Petersburg (former Leningrad Polytechnic University) by Prof. Anatolii I. Lurie. To my father Prof. Mikhail Vetyukov I am obliged for the decision to choose mechanics for my studies and further work. Prof. Hans Irschik from the Johannes Kepler University Linz has greatly contributed to the present work with his vivid interest to the subject and many important comments, which helped improving the quality and readability of the text. I also express my gratitude to Prof. Alexander Belyaev from the Polytechnic University of St. Petersburg, as well as to Prof. Michael Krommer, Dr. Peter Gruber, Dr. Alexander Humer and other colleagues from the Johannes Kepler University Linz for important discussions and for their attention to the manuscript. This work has been supported by the Austrian COMET-K2 programme of the Linz Center of Mechatronics (LCM), and was funded by the Austrian federal government and the federal state of Upper Austria. I am very thankful to my mother Olga and my daughters Anastasia and Elena, who have been a source of inspiration and encouragement in my life and in writing this book.

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Yury Vetyukov

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Chapter 1
Introduction

Abstract We begin with a brief discussion of mathematical methods, which to a large extent determine the success of the analysis of thin-walled structures: a compact and consistent notation for the invariant tensor calculus in the three-dimensional Euclidean space; the procedure of asymptotic splitting, which is proven to be efficient for the dimensional reduction in the theories of thin bodies; the principle of virtual work in application to continuum mechanics; variational methods as a basis for numerical applications. The state of the art in the mechanics of thin-walled structures is discussed on the example plane stress problem of bending of a straight strip. In the literature review, the past research in the field is classified into the method of hypotheses, variational approaches, direct approaches and asymptotic methods. The introduction is concluded with a discussion of a hybrid asymptotic–direct approach, which is applied throughout the book to various kinds of thin-walled structures.

1.1 Fundamentals: Analytical Technologies

The history of structural mechanics includes several important points, which influenced the agenda of research in this field. The solution of Saint-Venant for a prismatic rod, the use of variational approaches with approximations of the unknown displacements, strains and/or stresses over the thickness, the development of numerical methods broadened the spectrum of treatable problems and increased the trustworthiness of the results. Nevertheless, the theories of thin-walled structures are still often regarded as approximate engineering methods. And it means that the development is yet far from complete.

Recent advances in the analysis of the asymptotic behavior of exact solutions for thin bodies, as well as in the direct approach to dimensionally reduced continua allow speaking about a hybrid approach, which is discussed in the last section of this introductory chapter. This novel way of thinking requires particular mathematical methods, or, rather, analytical technologies, with which we begin the introduction, and which constitute a basis for the substantial part of the book.

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1.1.1 Invariant Vectors and Tensors in Space

Tensor calculus is inherent to the mechanics of deformable bodies. Here we briefly summarize the basics of the tensor algebra in three-dimensional Euclidean space, which allows for a certain simplification in comparison to the general notation, established in the mathematical literature.

Vectors in space are defined by their magnitude and direction. One can add vectors, multiply them with a scalar coefficient, or compute scalar and vector products. A system of Cartesian axes x_i, $i = 1, \ldots, 3$, determines a basis, which consists of three unit and orthogonal (orthonormal) vectors $\boldsymbol{e}_i$ with the scalar products

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \equiv \begin{cases} 1, & i = j, \\ 0, & i \neq j; \end{cases} \tag{1.1}$$

here the Kronecker symbol δ_{ij} is introduced.

A vector can be decomposed in this basis into components:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i \equiv a_i \boldsymbol{e}_i. \tag{1.2}$$

The Einstein convention is used: repeated indices imply summation. The scalar product of two vectors, which is commonly defined as a product of their magnitudes and of the cosine of the angle between their directions, can be expressed via components:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i. \tag{1.3}$$

Components of a vector are simply computed as $\boldsymbol{a} \cdot \boldsymbol{e}_i = a_k \boldsymbol{e}_k \cdot \boldsymbol{e}_i = a_k \delta_{ki} = a_i$.

Consider another Cartesian basis $\boldsymbol{e}'_i$; it is related to the original one $\boldsymbol{e}_i$ with the direction cosines

$$\alpha_{ik} \equiv \boldsymbol{e}'_i \cdot \boldsymbol{e}_k, \quad \boldsymbol{e}'_i = \alpha_{ik} \boldsymbol{e}_k. \tag{1.4}$$

The matrix $\{\alpha_{ik}\}$ is orthogonal:

$$\delta_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}'_j = \alpha_{ik} \alpha_{jn} \boldsymbol{e}_k \cdot \boldsymbol{e}_n = \alpha_{ik} \alpha_{jn} \delta_{kn} = \alpha_{ik} \alpha_{jk}; \tag{1.5}$$

only the terms with equal indices k and n remain in the sum because of δ_{kn}.

Invariant vectors are independent from the choice of the basis:

$$\boldsymbol{a} = a_i \boldsymbol{e}_i = a'_i \boldsymbol{e}'_i \quad \Rightarrow \quad a'_i = \alpha_{ik} a_k. \tag{1.6}$$

This law of transformation of components is traditionally used in mathematics as a definition of a vector as a tensor of the first rank: if in any basis we have a triple of values a_i, which obey (1.6), then an invariant object $\boldsymbol{a}$ is defined. The magnitude of a vector is its only scalar invariant:

$$|\boldsymbol{a}| \equiv \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_i a_i} = \sqrt{a'_i a'_i}. \tag{1.7}$$

Three different methodologies for dealing with vectors and tensors find use in the literature on continuum mechanics.

- A matrix (column of components) $\{a_i\}$ is often identified with the physical vector itself. With a due extension to higher rank tensors, this attractive approach is frequently used in the engineering literature and in computational applications, see, e.g., [26, 74, 140]. Both simplicity and convenience vanish in the geometrically nonlinear elasticity, when multiple oblique coordinate frames need to be considered simultaneously: the symbols a_i on their own are just three scalars, which are related to the invariant vector $\boldsymbol{a}$ only when the particular basis $\boldsymbol{e}_i$ is known. The most evident issue of the matrix notation is that each matrix of components shall be "accompanied" by a particular basis, which needs to be kept in mind.
- The so-called index (or coordinate) notation is used in continuum mechanics with a high level of consistency [18, 56, 123, 139, 155]. Analysis in curvilinear coordinates may lead to complicated expressions owing to the derivatives of the basis vectors.
- The direct tensor calculus [40, 96, 101, 103, 154] operates with invariant objects and is often opposed to the index notation. Fundamental equations of mechanics, which are not related to any particular basis, shall advantageously be written in an invariant form, but intermediate mathematical transformations may be difficult to perform.

In the present book the strong sides of both the index and the direct notations are combined in a manner proposed by Lurie [103]; see also Lebedev et al. [96], Eliseev [51] as well as the compact and comprehensive textbook by Danielson [40]. When it comes to numerical analysis, then a particular coordinate frame is chosen and the computations are performed using operations on matrices.

Tensors of a higher rank are commonly introduced according to a definition, similar to the one after (1.6). Thus, if in each basis we have nine values T_{ij} with the transformation law

$$T'_{ij} = \alpha_{ik}\alpha_{in}T_{kn}, \tag{1.8}$$

then an invariant object $\mathbf{T}$ is defined. The values T_{kn} are the components of this tensor of the second rank. Thus if both $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors (tensors of the first rank), i.e., if they fulfill (1.6), then it is easy to check that nine values $c_{ij} = a_i b_j$ follow from (1.8). The tensor

$$\mathbf{c} = \boldsymbol{ab} \tag{1.9}$$

is called a dyad; the symbol $\otimes$ is sometimes used to indicate the tensor (or dyadic) product. A dyadic product of three vectors produces a triad, which is a tensor of the third rank: ${}^3\mathbf{A} = \boldsymbol{abc}$, $A_{ijk} = a_i b_j c_k$.

The *identity tensor* $\mathbf{I}$ has components δ_{ij} in each Cartesian basis; (1.8) is fulfilled because of (1.5). An invariant linear mapping of a vector field into another one $\boldsymbol{a} \mapsto \boldsymbol{b}$ defines a tensor of the second rank: in each basis we have $a_i = c_{ij}b_j$, and the coefficients c_{ij} will obey (1.8).

Four basic operations may be performed on tensors. The first one, which combines summation and multiplication with a scalar, produces a *linear combination* of tensors of the same rank:

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} \quad \Rightarrow \quad c_{ij} = \lambda a_{ij} + \mu b_{ij}. \tag{1.10}$$

The second operation is the *tensor product*: (1.9) may be generalized to tensors of arbitrary rank, e.g.,

$$\boldsymbol{a}\mathbf{T} = {}^3\mathbf{A} \quad \Rightarrow \quad a_i T_{jk} = A_{ijk}. \tag{1.11}$$

The *contraction* is the third operation: summation over a pair of indices reduces the rank of a tensor by two. Contraction within a second rank tensor gives its first scalar invariant, which is called trace:

$$\operatorname{tr}\mathbf{T} \equiv T_{ii}. \tag{1.12}$$

Three invariant vectors can be produced by ${}^3\mathbf{A}$ by a contraction with respect to different pairs of indices:

$$A_{iik} = a_k, \quad A_{iji} = b_j, \quad A_{ijj} = c_i. \tag{1.13}$$

The *transposition* of a second rank tensor

$$\mathbf{A} = \mathbf{B}^T \quad \Rightarrow \quad A_{ij} = B_{ji} \tag{1.14}$$

is a particular case of *index permutation*, which is the fourth basic operation.

The *scalar product* is a combination of the tensor product with the contraction:

$$\boldsymbol{a}\cdot\mathbf{T} = \boldsymbol{b} \quad \Rightarrow \quad a_i T_{ij} = b_j, \quad \boldsymbol{a}\cdot\mathbf{T}\cdot\boldsymbol{b} = a_i T_{ij} b_j, \quad \mathbf{I}\cdot\mathbf{A} = \mathbf{A}. \tag{1.15}$$

Examples with the double contraction are

$$\mathbf{A}\cdot\cdot\mathbf{B} = A_{ij}B_{ji}, \quad {}^4\mathbf{C}\cdot\cdot\boldsymbol{\varepsilon} = \boldsymbol{\tau} \quad \Rightarrow \quad C_{ijkl}\varepsilon_{lk} = \tau_{ij}, \quad \mathbf{I}\cdot\cdot\mathbf{A} = \operatorname{tr}\mathbf{A}. \tag{1.16}$$

In the western-world literature on mechanics the double contraction is often denoted as ":" with a slightly different meaning, which is restricted to tensors of the second rank:

$$\mathbf{A} : \mathbf{B} = A_{ij}B_{ij} = \mathbf{A}\cdot\cdot\mathbf{B}^T. \tag{1.17}$$

A tensor is related to its components:

$$\begin{aligned} \mathbf{T} &= T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j, \quad T_{ij} = \boldsymbol{e}_i\cdot\mathbf{T}\cdot\boldsymbol{e}_j; \\ \boldsymbol{a}\cdot\mathbf{T} &= a_i\boldsymbol{e}_i\cdot T_{jk}\boldsymbol{e}_j\boldsymbol{e}_k = a_i T_{jk}\delta_{ij}\boldsymbol{e}_k = a_i T_{ik}\boldsymbol{e}_k. \end{aligned} \tag{1.18}$$

The *vector product* is a contraction with the *Levi-Civita tensor* ${}^3\boldsymbol{\epsilon}$:

$$\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{b}\boldsymbol{a}\cdot\cdot{}^3\boldsymbol{\epsilon}, \quad {}^3\boldsymbol{\epsilon} = \epsilon_{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k; \quad \epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i). \tag{1.19}$$

The Levi-Civita symbols are closely related to the notion of a *right-handed basis*, in which $\boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 1$:

$$\epsilon_{ijk} = \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k, \quad \boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k, \quad \mathbf{I} \times \mathbf{I} = \boldsymbol{e}_i\boldsymbol{e}_i \times \boldsymbol{e}_j\boldsymbol{e}_j = -{}^3\boldsymbol{\epsilon}. \qquad (1.20)$$

The rule of *cyclic permutation* in a triple, or mixed scalar and vector product reads

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a} \times \boldsymbol{b}; \qquad (1.21)$$

the order of operations is here uniquely defined.

For any symmetric second rank tensor $\mathbf{T} = \mathbf{T}^T$ one can find such an orthonormal basis $\boldsymbol{v}_i$, in which the components with different indices vanish:

$$\mathbf{T} = \sum_i \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i; \qquad (1.22)$$

the rule of summation over a repeating index cannot be applied in this basis. The eigenvectors $\boldsymbol{v}_i$ and the eigenvalues λ_i are determined by the *eigenvalue problem*

$$\mathbf{T} \cdot \boldsymbol{v} = \lambda \boldsymbol{v} \quad \Rightarrow \quad \det(\mathbf{T} - \lambda \mathbf{I}) = 0, \qquad (1.23)$$

which involves the notion of a *determinant* of a second rank tensor. In Cartesian components we deal with a common matrix formulation:

$$T_{ij} v_j = \lambda v_i \quad \Rightarrow \quad \det\{T_{ij} - \lambda\delta_{ij}\} = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0. \qquad (1.24)$$

The coefficients of the cubic characteristic equation for the eigenvalues λ are called principal invariants of a tensor, and $I_1(\mathbf{T}) = \operatorname{tr}\mathbf{T}$, $I_3(\mathbf{T}) = \det\mathbf{T}$.

The eigenvalues are real numbers and the eigenvectors (which answer to different eigenvalues) are orthogonal provided that the tensor is symmetric. With the representation (1.22) one can compute an arbitrary power of the tensor, e.g.,

$$\mathbf{T}^2 \equiv \mathbf{T} \cdot \mathbf{T} = \sum_i \lambda_i^2 \boldsymbol{v}_i \boldsymbol{v}_i, \quad \mathbf{T}^{1/2} = \sum_i \lambda_i^{1/2} \boldsymbol{v}_i \boldsymbol{v}_i, \qquad (1.25)$$

which allows to extend the notion of series expansions of analytic functions to the case of tensorial arguments.

Any second rank tensor is a sum of its *symmetric* and *antisymmetric* parts:

$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A, \quad \mathbf{T}^S \equiv \frac{1}{2}(\mathbf{T} + \mathbf{T}^T), \quad \mathbf{T}^A \equiv \frac{1}{2}(\mathbf{T} - \mathbf{T}^T). \qquad (1.26)$$

A *skew-symmetric* (or antisymmetric) tensor

$$\mathbf{B} = -\mathbf{B}^T, \quad B_{ij} = -B_{ji} \qquad (1.27)$$

can be expressed through the associated vector $\boldsymbol{b}$, which is related to its vector invariant $\boldsymbol{B}_\times$ (which is sometimes called "Gibbsian cross" [96]):

$$\mathbf{B} = \boldsymbol{b} \times \mathbf{I} = \mathbf{I} \times \boldsymbol{b}, \quad \boldsymbol{b} = -\frac{1}{2}\boldsymbol{B}_\times, \quad \boldsymbol{B}_\times \equiv B_{ij}\boldsymbol{e}_i \times \boldsymbol{e}_j. \qquad (1.28)$$

Indeed,

$$\begin{aligned}\boldsymbol{b}\times\mathbf{I} &= -\frac{1}{2}B_{ij}(\boldsymbol{e}_i\times\boldsymbol{e}_j)\times\boldsymbol{e}_k\boldsymbol{e}_k\\ &= -\frac{1}{2}B_{ij}(\boldsymbol{e}_j\delta_{ik}-\boldsymbol{e}_i\delta_{jk})\boldsymbol{e}_k=\frac{1}{2}\big(\mathbf{B}-\mathbf{B}^T\big)=\mathbf{B}.\end{aligned} \tag{1.29}$$

Here the known equality for a *double vector product* is applied:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\boldsymbol{a}\cdot\boldsymbol{c}-\boldsymbol{c}\boldsymbol{a}\cdot\boldsymbol{b}. \tag{1.30}$$

Functions of tensor and vector arguments require an invariant definition: a mapping $\Phi(\mathbf{T})$ exists in any basis as an equivalent function of the components $\Phi(T_{ij})$ such that the value remains independent from the basis. An invariant scalar function of a second rank tensor depends only on its three invariants. A derivative of a function is defined according to

$$\mathrm{d}\Phi=\frac{\partial\Phi}{\partial\mathbf{T}}\cdot\cdot\,\mathrm{d}\mathbf{T}^T,\quad \frac{\partial\Phi}{\partial\mathbf{T}}=\frac{\partial\Phi}{\partial T_{ij}}\boldsymbol{e}_i\boldsymbol{e}_j. \tag{1.31}$$

A *rotation tensor* connects two bases:

$$\boldsymbol{e}_i'=\mathbf{P}\cdot\boldsymbol{e}_i,\quad \mathbf{P}=\boldsymbol{e}_i'\boldsymbol{e}_i=\alpha_{ij}\boldsymbol{e}_j\boldsymbol{e}_i;\quad \mathbf{P}\cdot\mathbf{P}^T=\mathbf{I},\quad \det\mathbf{P}=1. \tag{1.32}$$

According to Euler's theorem, for any rotation we can find its axis $\boldsymbol{k}$ (which is a unit vector, $\boldsymbol{k}\cdot\boldsymbol{k}=1$) and angle of rotation θ, which provides an invariant form for the rotation tensor [40, 103]:

$$\mathbf{P}=\mathbf{Q}(\theta,\boldsymbol{k})=\mathbf{I}\cos\theta+\boldsymbol{k}\times\mathbf{I}\sin\theta+\boldsymbol{k}\boldsymbol{k}(1-\cos\theta). \tag{1.33}$$

The variation of a rotation tensor is skew-symmetric:

$$\delta\big(\mathbf{P}\cdot\mathbf{P}^T\big)=0\quad\Rightarrow\quad\delta\mathbf{P}\cdot\mathbf{P}^T=-\big(\delta\mathbf{P}\cdot\mathbf{P}^T\big)^T=\delta\boldsymbol{\theta}\times\mathbf{I}\quad\Rightarrow\quad\delta\mathbf{P}=\delta\boldsymbol{\theta}\times\mathbf{P}, \tag{1.34}$$

in which $\delta\boldsymbol{\theta}$ is a small rotation vector. It is important to notice that the vector of small rotation shall not be considered as a variation of a stand-alone vector $\boldsymbol{\theta}$. There exist various finite rotation vectors, but none of them fulfills (1.34). Thus, it is easy to check that $\delta\boldsymbol{\theta}\neq\delta(\boldsymbol{k}\theta)$.

Working with large deformations of solid bodies requires the notion of an *oblique basis*. We consider three linear independent vectors $\boldsymbol{a}_i$. A reciprocal basis (or cobasis) $\boldsymbol{a}^j$ needs to be introduced according to

$$\boldsymbol{a}_i\cdot\boldsymbol{a}^j=\delta_i^j. \tag{1.35}$$

Any vector can be represented with two types of components:

$$\boldsymbol{v}=v^i\boldsymbol{a}_i=v_i\boldsymbol{a}^i,\quad v^i=\boldsymbol{v}\cdot\boldsymbol{a}^i,\quad v_i=\boldsymbol{v}\cdot\boldsymbol{a}_i. \tag{1.36}$$

The summation is performed on one upper and one lower index, and free (non-repeating) indices should be on the same level at both sides of an equality. The components v_i are denoted as covariant, and v^i as contravariant. A second rank tensor has four different types of components:

$$\mathbf{T} = T_{ij}\boldsymbol{a}^i\boldsymbol{a}^j = T^{ij}\boldsymbol{a}_i\boldsymbol{a}_j = T_i^{\cdot j}\boldsymbol{a}^i\boldsymbol{a}_j = T^i_{\cdot j}\boldsymbol{a}_i\boldsymbol{a}^j,$$
$$T_{ij} = \boldsymbol{a}_i \cdot \mathbf{T} \cdot \boldsymbol{a}_j, \quad T_i^{\cdot j} = \boldsymbol{a}_i \cdot \mathbf{T} \cdot \boldsymbol{a}^j, \quad \ldots. \tag{1.37}$$

Mixed components of the identity tensor form an identity matrix, and the matrices of its co- and contravariant components are mutually inverse:

$$\mathbf{I} = g_{ij}\boldsymbol{a}^i\boldsymbol{a}^j = g^{ij}\boldsymbol{a}_i\boldsymbol{a}_j = \boldsymbol{a}_i\boldsymbol{a}^i = \boldsymbol{a}^i\boldsymbol{a}_i,$$
$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j, \quad g^{ij} = \boldsymbol{a}^i \cdot \boldsymbol{a}^j, \quad g_i^{\cdot j} = \delta_i^j, \quad g^{ij}g_{jk} = \delta_k^i. \tag{1.38}$$

Components g_{ij} or g^{ij} are often called "components of a metric tensor" and allow raising and lowering the indices:

$$\boldsymbol{a}_i = g_{ij}\boldsymbol{a}^j, \quad \boldsymbol{a}^i = g^{ij}\boldsymbol{a}_j, \quad v_i = g_{ij}v^j, \quad v^i = g^{ij}v_j. \tag{1.39}$$

In contrast to (1.24), now the habitual expressions of invariants keep working only with the matrices of mixed components:

$$\det \mathbf{T} = \det\{T_i^{\cdot j}\} = \det\{T^i_{\cdot j}\}. \tag{1.40}$$

A point in the three-dimensional space is identified by its *place*, or *position vector* $\boldsymbol{r} = x_i\boldsymbol{e}_i$. A *scalar field* $u(\boldsymbol{r})$ can be described in each Cartesian basis as a function of three arguments $u(x_i)$. Consider its differential:

$$\mathrm{d}u = \partial_i u \, \mathrm{d}x_i = \mathrm{d}\boldsymbol{r} \cdot \nabla u, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \nabla \equiv \boldsymbol{e}_i \partial_i. \tag{1.41}$$

Here the *invariant differential operator* ∇ (Hamilton's operator) produces an invariant vector ∇u, which is the gradient of the scalar field. The equality $\mathrm{d}u = \mathrm{d}\boldsymbol{r} \cdot \nabla u$ shall be considered as a definition of ∇.

For a *vector field* $\boldsymbol{v}(\boldsymbol{r})$ the gradient is a tensor

$$\operatorname{grad} \boldsymbol{v} \equiv \nabla \boldsymbol{v} = \boldsymbol{e}_i \partial_i \boldsymbol{v} = \boldsymbol{e}_i \boldsymbol{e}_j \partial_i v_j, \quad \mathrm{d}\boldsymbol{v} = \mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{v}. \tag{1.42}$$

The trace and the vector invariant of $\nabla \boldsymbol{v}$ are correspondingly the *divergence* and the *curl* (rotor) of the vector field:

$$\operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v} = \partial_i v_i, \quad \operatorname{rot} \boldsymbol{v} \equiv \nabla \times \boldsymbol{v} = \epsilon_{ijk} \partial_i v_j \boldsymbol{e}_k. \tag{1.43}$$

Integral theorems of the field theory [40, 145] play an important role in continuum mechanics. The divergence, or the Gauss–Ostrogradsky flux theorem states that the total flux of a vector (or tensor) field $\boldsymbol{v}$ through a closed surface $\Omega = \partial V$ equals the total divergence of the field in the volume V within the surface:

$$\int_\Omega \boldsymbol{n}\cdot\boldsymbol{v}\,\mathrm{d}\Omega = \int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V; \tag{1.44}$$

$\boldsymbol{n}$ is the vector of outer unit normal.

The curl (or Stokes) theorem

$$\oint_{\partial\Omega} (\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{v}) = \int_\Omega \boldsymbol{n}\cdot\nabla\times\boldsymbol{v}\,\mathrm{d}\Omega \tag{1.45}$$

relates the flux of the curl of a field through a surface Ω and the circulation of the field along the closed boundary contour $\partial\Omega$; the direction of the unit normal on the surface $\boldsymbol{n}$ corresponds to the direction along the contour $\mathrm{d}\boldsymbol{r}$ according to the right-hand rule.

The invariant definition of the differential operator remains the same in the arbitrary case of *curvilinear coordinates* q^i. The position vector of an arbitrary point is $\boldsymbol{r}(q^i)$. The vectors of derivatives $\boldsymbol{r}_i \equiv \partial_i \boldsymbol{r}$ constitute a basis, and the differential operator is written with the cobasis $\boldsymbol{r}^i$:

$$\nabla = \boldsymbol{r}^i\partial_i, \quad \mathrm{d}\boldsymbol{r}\cdot\nabla u = \boldsymbol{r}_i\cdot\boldsymbol{r}^k\partial_k u\,\mathrm{d}q^i = \partial_i u\,\mathrm{d}q^i = \mathrm{d}u. \tag{1.46}$$

The direct tensor calculus helps avoiding covariant and contravariant derivatives, metric components, Christoffel symbols and other attributes of the index notation [56]. But intermediate transformations are often easier with components:

$$\nabla\cdot(\boldsymbol{r}\boldsymbol{r}) = \boldsymbol{e}_i\cdot\partial_i(\boldsymbol{r}\boldsymbol{r}) = \boldsymbol{e}_i\cdot\boldsymbol{e}_i\boldsymbol{r} + \boldsymbol{e}_i\cdot\boldsymbol{r}\boldsymbol{e}_i = 3\boldsymbol{r} + \boldsymbol{r} = 4\boldsymbol{r}; \tag{1.47}$$

a computation with an arbitrary basis would lead to the same result, although the effort is minimal with the Cartesian basis.

1.1.2 Procedure of Asymptotic Splitting

Dealing with complicated problems, one can often assume certain quantities to be small. Formal asymptotic expansions in terms of a presumably *small parameter* help finding dominating effects in the solution. Consider an equation

$$g(u,\lambda) = 0 \tag{1.48}$$

with $\lambda \to 0$. We seek the solution in the form of a *series expansion* in terms of λ:

$$u = \overset{0}{u} + \lambda\overset{1}{u} + \lambda^2\overset{2}{u} + \cdots. \tag{1.49}$$

Having substituted (1.49) into Eq. (1.48), we expand the left-hand side into series with respect to the small parameter and equate the coefficients of like powers of