

# 有限元方法 基础理论

第6版

## THE FINITE ELEMENT METHOD ITS BASIS & FUNDAMENTALS



O.C. ZIENKIEWICZ, R.L. TAYLOR & J.Z. ZHU



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# **The Finite Element Method: Its Basis and Fundamentals**

**Sixth edition**

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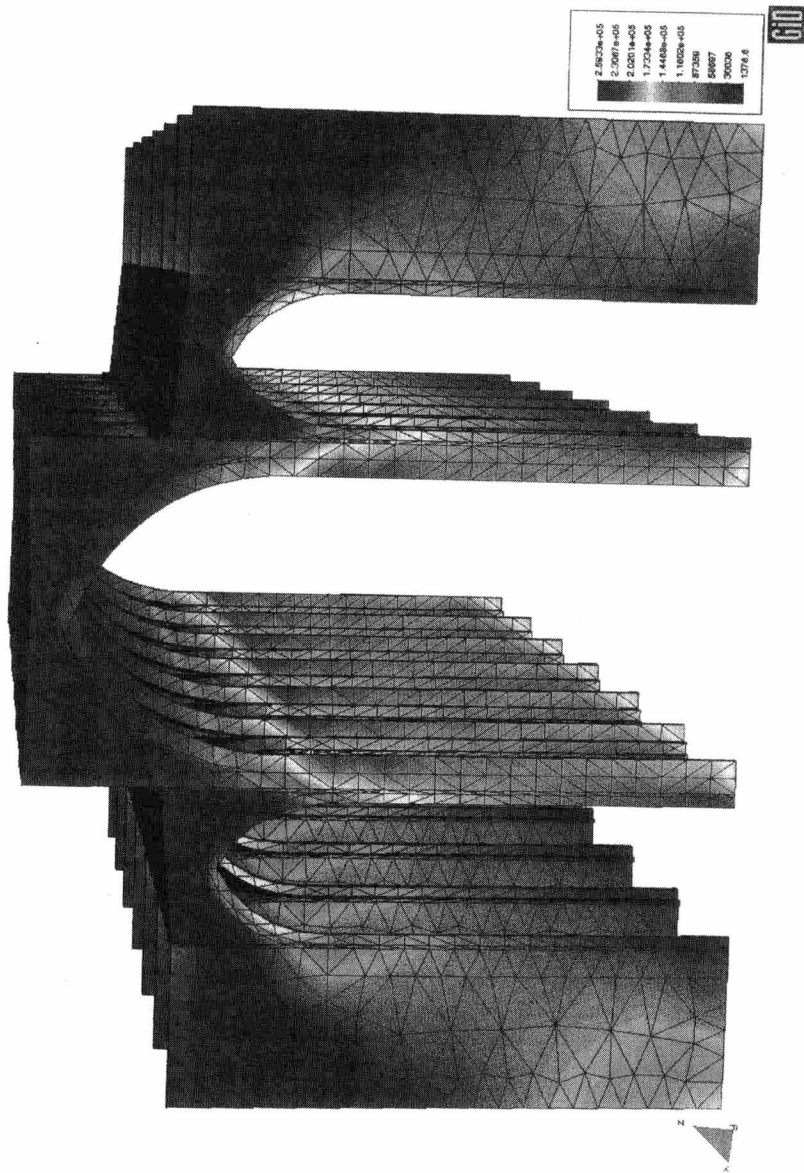
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**The Finite Element Method: Its  
Basis and Fundamentals  
Sixth edition**

**Professor O.C. Zienkiewicz, CBE, FRS, FREng** is Professor Emeritus at the Civil and Computational Engineering Centre, University of Wales Swansea and previously Director of the Institute for Numerical Methods in Engineering at the University of Wales Swansea, UK. He holds the UNESCO Chair of Numerical Methods in Engineering at the Technical University of Catalunya, Barcelona, Spain. He was the head of the Civil Engineering Department at the University of Wales Swansea between 1961 and 1989. He established that department as one of the primary centres of finite element research. In 1968 he became the Founder Editor of the *International Journal for Numerical Methods in Engineering* which still remains today the major journal in this field. The recipient of 27 honorary degrees and many medals, Professor Zienkiewicz is also a member of five academies – an honour he has received for his many contributions to the fundamental developments of the finite element method. In 1978, he became a Fellow of the Royal Society and the Royal Academy of Engineering. This was followed by his election as a foreign member to the U.S. Academy of Engineering (1981), the Polish Academy of Science (1985), the Chinese Academy of Sciences (1998), and the National Academy of Science, Italy (Accademia dei Lincei) (1999). He published the first edition of this book in 1967 and it remained the only book on the subject until 1971.

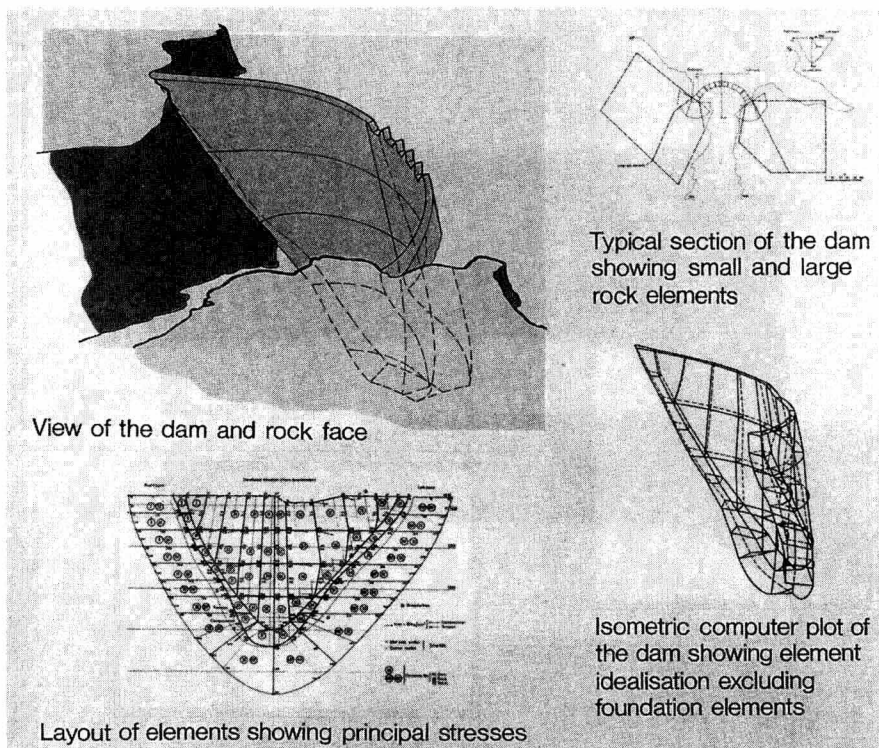
**Professor R.L. Taylor** has more than 40 years' experience in the modelling and simulation of structures and solid continua including two years in industry. He is Professor in the Graduate School and the Emeritus T.Y. and Margaret Lin Professor of Engineering at the University of California at Berkeley. In 1991 he was elected to membership in the US National Academy of Engineering in recognition of his educational and research contributions to the field of computational mechanics. Professor Taylor is a Fellow of the US Association of Computational Mechanics – USACM (1996) and a Fellow of the International Association of Computational Mechanics – IACM (1998). He has received numerous awards including the Berkeley Citation, the highest honour awarded by the University of California at Berkeley, the USACM John von Neumann Medal, the IACM Gauss–Newton Congress Medal and a Dr.-Ingenieur ehrenhalber awarded by the Technical University of Hannover, Germany. Professor Taylor has written several computer programs for finite element analysis of structural and non-structural systems, one of which, *FEAP*, is used world-wide in education and research environments. A personal version, *FEAPpv*, available from the publisher's website, is incorporated into the book.

**Dr J.Z. Zhu** has more than 20 years' experience in the development of finite element methods. During the last 12 years he has worked in industry where he has been developing commercial finite element software to solve multi-physics problems. Dr Zhu read for his Bachelor of Science degree at Harbin Engineering University and his Master of Science at Tianjin University, both in China. He was awarded his doctoral degree in 1987 from the University of Wales Swansea, working under the supervision of Professor Zienkiewicz. Dr Zhu is the author of more than 40 technical papers on finite element methods including several on error estimation and adaptive automatic mesh generation. These have resulted in his being named in 2000 as one of the highly cited researchers for engineering in the world and in 2001 as one of the top 20 most highly cited researchers for engineering in the United Kingdom.



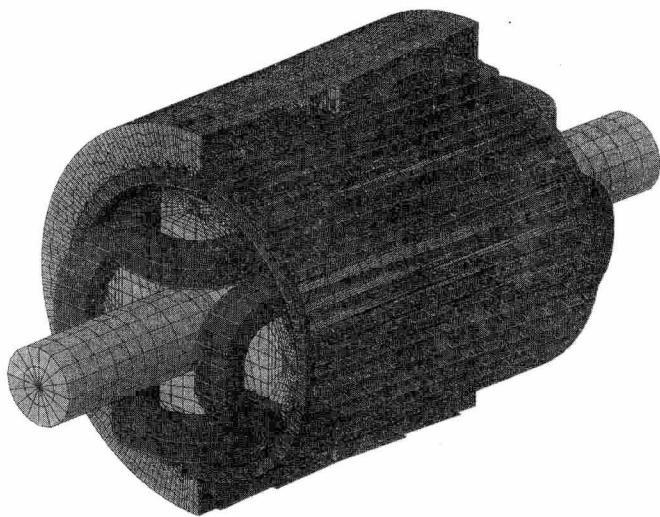
**Plate 1** Contours of the deviatoric stress invariants for the nave of Gothic cathedral

Courtesy of Prof. Miguel Cervera, CIMNE, Barcelona. Source: P. Roca, M. Cervera, L. Pellegrini and J. Torrent, 'Studies on the Structure of Gothic Cathedrals', Int. Conf. 40th Anniversary of the Int. Ass. Shells and Spatial Structures, Madrid, Spain 1999.

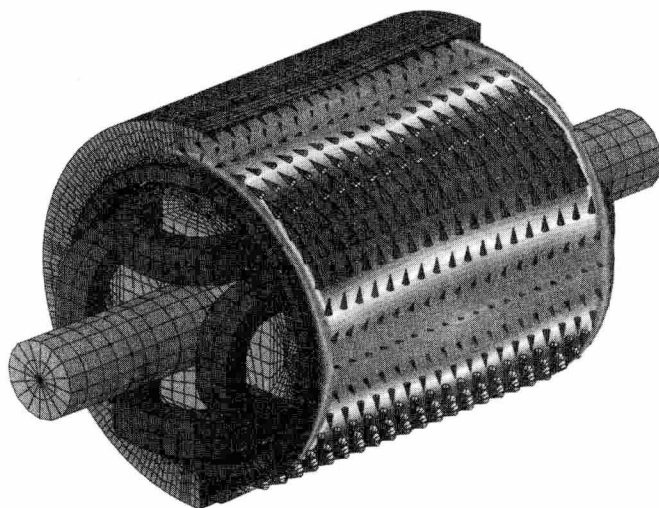


**Plate 2 Analysis of an arch dam in China**

An early three dimensional analysis (1970). Analysis by OCZ and Cedric Taylor, Department of Civil Engineering, University of Wales Swansea.



(a) Stator slots have been 'skewed' by one slot pitch in order to reduce cogging torque, i.e. torque associated with the alignment and misalignment of rotor poles and stator teeth.



(b) Contours and vectors that indicate the strength and direction of the magnetic fields in the stator core back.

### Plate 3 4-pole generator

Courtesy of Mr. William Trowbridge, Vector Fields, Kidlington, Oxfordshire. Source: J. Simkin and C.W. Trowbridge 'Three dimensional nonlinear electromagnetic field computations, using scalar potentials', *Proc. IEE*, **127**, Pt. B, No. 6, Nov. 1980.





**Plate 4** Scattering of a plane electromagnetic wave by a perfectly conducting aircraft

Courtesy of Prof. Ken Morgan, Department of Civil Engineering, University of Wales Swansea. Source: K. Morgan, P. J. Brookes, O. Hassan and N. P. Wetherill, 'Parallel processing for the simulation of problems involving scattering of electromagnetic waves, *Comp. Meth. Appl. Mech. Eng.*, **152**, 157-174, 1998.

# Preface

It is thirty-eight years since the *The Finite Element Method in Structural and Continuum Mechanics* was first published. This book, which was the first dealing with the finite element method, provided the basis from which many further developments occurred. The expanding research and field of application of finite elements led to the second edition in 1971, the third in 1977, the fourth as two volumes in 1989 and 1991 and the fifth as three volumes in 2000. The size of each of these editions expanded geometrically (from 272 pages in 1967 to the fifth edition of 1482 pages). This was necessary to do justice to a rapidly expanding field of professional application and research. Even so, much filtering of the contents was necessary to keep these editions within reasonable bounds.

In the present edition we have decided not to pursue the course of having three contiguous volumes but rather we treat the whole work as an assembly of three separate works, each one capable of being used without the others and each one appealing perhaps to a different audience. Though naturally we recommend the use of the whole ensemble to people wishing to devote much of their time and study to the finite element method.

In particular the first volume which was entitled *The Finite Element Method: The Basis* is now renamed *The Finite Element Method: Its Basis and Fundamentals*. This volume has been considerably reorganized from the previous one and is now, we believe, better suited for teaching fundamentals of the finite element method. The sequence of chapters has been somewhat altered and several examples of worked problems have been added to the text. A set of problems to be worked out by students has also been provided.

In addition to its previous content this book has been considerably enlarged by including more emphasis on use of higher order shape functions in formulation of problems and a new chapter devoted to the subject of automatic mesh generation. A beginner in the finite element field will find very rapidly that much of the work of solving problems consists of preparing a suitable mesh to deal with the whole problem and as the size of computers has seemed to increase without limits the size of problems capable of being dealt with is also increasing. Thus, meshes containing sometimes more than several million nodes have to be prepared with details of the material interfaces, boundaries and loads being well specified. There are many books devoted exclusively to the subject of mesh generation but we feel that the essence of dealing with this difficult problem should be included here for those who wish to have a complete 'encyclopedic' knowledge of the subject.

The chapter on computational methods is much reduced by transferring the computer source program and user instructions to a web site.† This has the very substantial advantage of not only eliminating errors in program and manual but also in ensuring that the readers have the benefit of the most recent version of the program available at all times.

The two further volumes form again separate books and here we feel that a completely different audience will use them. The first of these is entitled *The Finite Element Method in Solid and Structural Mechanics* and the second is a text entitled *The Finite Element Method in Fluid Dynamics*. Each of these two volumes is a standalone text which provides the full knowledge of the subject for those who have acquired an introduction to the finite element method through other texts. Of course the viewpoint of the authors introduced in this volume will be continued but it is possible to start at a different point.

We emphasize here the fact that all three books stress the importance of considering the finite element method as a unique and whole basis of approach and that it contains many of the other numerical analysis methods as special cases. Thus, imagination and knowledge should be combined by the readers in their endeavours.

The authors are particularly indebted to the International Center of Numerical Methods in Engineering (CIMNE) in Barcelona who have allowed their pre- and post-processing code (GiD) to be accessed from the web site. This allows such difficult tasks as mesh generation and graphic output to be dealt with efficiently. The authors are also grateful to Professors Eric Kasper and Jose Luis Perez-Aparicio for their careful scrutiny of the entire text and Drs Joaquim Peiró and C.K. Lee for their review of the new chapter on mesh generation.

### ***Resources to accompany this book***

Worked solutions to selected problems in this book are available online for teachers and lecturers who either adopt or recommend the text. Please visit <http://books.elsevier.com/manuals> and follow the registration and log in instructions on screen.

OCZ, RLT and JZZ

† Complete source code and user manual for program *FEAPpv* may be obtained at no cost from the publisher's web page: <http://books.elsevier.com/companions> or from the authors' web page: <http://www.ce.berkeley.edu/~rlt>

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