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BRANCHING PROCESSES, BRANCHING RANDOM WALKS AND BRANCHING PARTICLE FIELDS SET

Volume 1

Discrete Time Branching Processes in Random Environment

Götz Kersting
Vladimir Vatutin

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BRANCHING PROCESSES, BRANCHING RANDOM WALKS AND BRANCHING PARTICLE FIELDS SET
Coordinated by Elena Yarovaya

Branching processes are stochastic processes which represent the reproduction of particles, such as individuals within a population, and thereby model demographic stochasticity. In branching processes in random environment (BPREs), additional environmental stochasticity is incorporated, meaning that the conditions of reproduction may vary in a random fashion from one generation to the next.

The fundamental questions conform to those dealt with in general branching processes, such as supercritical, critical and subcritical behavior, the asymptotic size of the population's extinction probability and so on. Yet compared to other branching processes BPREs exhibit unique features, since their properties are mainly determined by the environmental randomness. This goes along with rather different methods of investigation resting largely on asymptotic results from the theory of random walks.

This book offers an introduction to the basics of BPREs and then presents the cases of critical and subcritical processes in detail, the latter dividing into weakly, intermediate, and strongly subcritical regimes.

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Discrete Time Branching Processes in Random Environment

Preface

Branching processes constitute a fundamental part in the theory of stochastic processes. Very roughly speaking, the theory of branching processes deals with the issue of exponential growth or decay of random sequences or processes. Its central concept consists of a system or a population made up of particles or individuals which independently produce descendants. This is an extensive topic dating back to a publication of F. Galton and H. W. Watson in 1874 on the extinction of family names and afterwards dividing into many subareas. Correspondingly, it is treated in a number of monographs starting in 1963 with T. E. Harris' seminal *The Theory of Branching Processes* and supported in the middle of the 1970s by Sevastyanov's *Verzweigungsprozesse*, Athreya and Ney's *Branching Processes*, Jagers' *Branching Processes with Biological Applications* and others. The models considered in these books mainly concern branching processes evolving in a constant environment. Important tools in proving limit theorems for such processes are generating functions, renewal type equations and functional limit theorems.

However, these monographs rarely touch the matter of branching processes in a random environment (BPREs). These objects form not so much a subclass but rather an extension of the area of branching processes. In such models, two types of stochasticity are incorporated: on the one hand, demographic stochasticity resulting from the reproduction of individuals and, on the other hand, environmental stochasticity stemming from the changes in the conditions of reproduction along time. A central insight is that it is often the latter component that primarily determines the behavior of these processes. Thus, the theory of BPREs gains its own characteristic appearance and novel aspects appear such as a phase transition. From a technical point of view, the study of such processes requires an extension of the range of methods to be used in comparison with the methods common in the classical theory of branching processes. Other techniques should be attracted, in particular from the theory of random walks which plays an essential role in proving limit theorems for BPREs.

With this volume, we have two purposes in mind. First, we have to assert that the basics of the theory of BPREs are somewhat scattered in the literature (since the late 1960s), from which they are not at all easily accessible. Thus, we start by presenting them in a unified manner. In order to simplify matters, we confine ourselves from the beginning to the case where the environment varies in an i.i.d. fashion, the model going back to the now classical paper written by Smith and Wilkinson in 1969. We also put together that material which is required from the topic of branching processes in a varying environment. Overall, the proofs are now substantially simplified and streamlined but, at the same time, some of the theorems could be better shaped.

Second, we would like to advance some scientific work on branching processes in a random environment conditioned on survival which was conducted since around 2000 by a German-Russian group of scientists consisting of Valery Afanasyev, Christian Böinghoff, Elena Dyakonova, Jochen Geiger, Götz Kersting, Vladimir Vatutin and Vitali Wachtel. This research was generously supported by the German Research Association DFG and the Russian Foundation for Basic Research RFBR. In this book, we again do not aim to present our results in their most general setting, yet an ample amount of technical context cannot be avoided.

We start the book by describing in Chapter 1 some properties of branching processes in a varying environment (BPVEs). In particular, we give a (short) proof of the theorem describing the necessary and sufficient conditions for the dichotomy in the asymptotic behavior of a BPVE: such a process should either die or its population size should tend to infinity with time. Besides the construction of family trees, size-biased trees and Geiger's tree, representing conditioned family trees, are described here in detail. These trees play an important role in studying subcritical BPREs. Chapter 2 leads the reader in to the world of BPREs. It contains classification of BPREs, describes some properties of supercritical BPREs and gives rough estimates for the growth rate of the survival probability for subcritical BPREs. Conclusions of Chapter 2 are supported by Chapter 3 where the asymptotic behavior of the probabilities of large deviations for all types of BPREs is investigated.

Properties of BPREs are closely related to the properties of the so-called associated random walk (ARW) constituted by the logarithms of the expected population sizes of particles of different generations. This justifies the appearance of Chapter 4 that includes some basic results on the theory of random walks and a couple of findings concerning properties of random walks conditioned to stay non-negative or negative and probabilities of large deviations for different types of random walks.

Chapters 5 through 9 deal with various statements describing the asymptotic behavior of the survival probability and Yaglom-type functional conditional limit theorems for the critical and subcritical BPREs and analyzing properties of the ARW providing survival of a BPRE for a long time. Here, the theory of random walks conditioned to stay non-negative or negative demonstrates its beauty and power.

Thus, it is shown in Chapter 5 (under the annealed approach) that if a critical BPRE survives up to a distant moment n, then the minimum value of the ARW on the interval $[0, n]$ is attained at the beginning of the evolution of the BPRE and the longtime behavior of the population size of such BPREs (conditioned on survival) resembles the behavior of the ordinary supercritical Galton–Watson branching processes. If, however, a critical BPRE is considered under the quenched approach (Chapter 6) then, given the survival of the process for a long time, the evolution of the population size in the past has an oscillating character: the periods when the population size was very big were separated by intervals when the size of the population was small.

Chapters 7–9 are devoted to the weakly, intermediately and strongly subcritical BPREs investigated under the annealed approach. To study properties of such processes, it is necessary to make changes in the initial measures based on the properties of the ARWs. The basic conclusion of Chapters 7–8 is: the survival probability of the weakly and intermediately subcritical BPREs up to a distant moment n is proportional to the probability for the corresponding ARW to stay non-negative within the time interval $[0, n]$. Finally, it is shown in Chapter 9 that properties of strongly subcritical BPREs are, in many respect, similar to the properties of the subcritical Galton–Watson branching processes. In particular, the survival probability of such a process up to a distant moment n is proportional to the expected number of particles in the process at this moment.

We do not pretend that this book includes all interesting and important results established up to now for BPREs. In particular, we do not treat here BPREs with immigration and multitype BPREs. The last direction of the theory of BPREs is a very promising field of investigation that requires study of properties of Markov chains generated by products of random matrices. To attract the attention of future researchers to this field, we give a short survey of some recent results for multitype BPREs in Chapter 10. The book is concluded by an Appendix that contains statements of results used in the proofs of some theorems but not fitting the main line of the monograph.

Götz KERSTING
Vladimir VATUTIN
August 2017

List of Notations

$\mathcal{P}(\mathbb{N}_0)$	set of all probability measures f on $\mathbb{N}_0 = \{0, 1, \ldots\}$, 2
$f[z]$, $f(s)$	weights and generating function of the measure f, 2
$\bar{f}$	the mean of the measure f, 2
$\tilde{f}$	the normalized second factorial moment, 2
f^*	size-biased measure, 21
$f_{m,n} := f_{m+1} \circ \cdots \circ f_n$	convolutions of probability measures, 3
$f_{m,n} := f_m \circ \cdots \circ f_{n+1}$	convolutions of probability measures, 107
$\varkappa(a, f) := (\bar{f}\,)^{-2} \sum_{y=a}^{\infty} y^2 f[y]$	truncated second moment, 102
θ	the moment of extinction of a branching process, 6, 28
T^*	size-biased tree, 21
$\mathcal{V}$	random environment, 25
$X := \log \overline{F}$	logarithm of expected population size, 27
$\kappa(\lambda) := \log \mathbb{E}[e^{\lambda X}]$	cumulant generating function, 36
γ	strict descending ladder epoch, 61
γ'	weak descending ladder epoch, 61
Γ	strict ascending ladder epoch, 61
Γ'	weak ascending ladder epoch, 62
$L_n := \min(S_0, S_1, \ldots, S_n)$	38
$M_n := \max(S_1, \ldots, S_n)$	76
$\tau(n) := \min\{0 \leq k \leq n : S_k = L_n\}$	the left-most moment when the minimum value of the random walk on the interval $[0, n]$ is attained, 70
$\mathbf{P}$	probability measure given the environment, 3, 26
$\mathbf{E}$	expectation given the environment, 3, 26

$\mathbb{P}$	probability measure obtained by averaging with respect to the environment, 25
$\mathbb{E}$	expectation taken after averaging with respect to the environment, 25
$\mathbb{P}^{+}, \mathbb{P}^{-}$	change of measure, 93
$\mathbb{P}^{\pm}$	change of measure, 134
$\widehat{\mathbb{P}}$	change of measure, 170, 198
$\widehat{\mathbb{P}}^{+}, \widehat{\mathbb{P}}^{-}$	change of measure, 172
$\mathbb{P}^{*}$	change of measure, 234

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