Grundlehren der mathematischen Wissenschaften 251

Shui-Nee Chow Jack K. Hale

Methods of Bifurcation Theory

Shui-Nee Chow Jack K. Hale

Methods of Bifurcation Theory

With 97 Illustrations

Springer-Verlag
New York Berlin Heidelberg

Shui-Nee Chow
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
U.S.A.

Jack K. Hale
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912
U.S.A.

AMS Subject Classifications 34Bxx, 34Cxx, 34Kxx, 35Bxx, 47A55, 47H15, 47H17, 58Cxx, 58Exx

Library of Congress Cataloging in Publication Data

Chow, Shui-Nee.
Methods of bifurcation theory.
(Grundlehren der mathematischen Wissenschaften; 251)
Bibliography: p
Includes index.
1. Functional differential equations. 2. Bifurcation theory. 3. Manifolds (Mathematics) I. Hale, Jack K. II. Title. III. Series.
QA372.C544 515.3'5 81-23337 AACR2

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90664-9 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90664-9 Springer-Verlag Berlin Heidelberg New York

Preface

An alternative title for this book would perhaps be Nonlinear Analysis, Bifurcation Theory and Differential Equations. Our primary objective is to discuss those aspects of bifurcation theory which are particularly meaningful to differential equations.

To accomplish this objective and to make the book accessible to a wider audience, we have presented in detail much of the relevant background material from nonlinear functional analysis and the qualitative theory of differential equations. Since there is no good reference for some of the material, its inclusion seemed necessary.

Two distinct aspects of bifurcation theory are discussed—static and dynamic. Static bifurcation theory is concerned with the changes that occur in the structure of the set of zeros of a function as parameters in the function are varied. If the function is a gradient, then variational techniques play an important role and can be employed effectively even for global problems. If the function is not a gradient or if more detailed information is desired, the general theory is usually local. At the same time, the theory is constructive and valid when several independent parameters appear in the function. In differential equations, the equilibrium solutions are the zeros of the vector field. Therefore, methods in static bifurcation theory are directly applicable.

Dynamic bifurcation theory is concerned with the changes that occur in the structure of the limit sets of solutions of differential equations as parameters in the vector field are varied. For example, in addition to discussing the way that the set of zeros of the vector field (the equilibrium solutions) change through the static theory, the stability properties of these solutions must be considered. In fact, there is an intimate relationship between changes of stability and bifurcation. The dynamics in a differential equation can also introduce other types of bifurcations; for example, periodic orbits, homoclinic orbits, invariant tori. This introduces several difficulties which require rather advanced topics from differential equations for their resolution.

The introductory chapter is designed to acquaint the reader with some of the types of problems that occur in bifurcation theory. The tools from nonlinear functional analysis are presented in Chapter 2. Some of this material is used more extensively in the text than others, but all topics are a necessary part of the vocabulary of persons working in bifurcation theory.

Some of the presentations and details of proofs are different from standard ones.

Chapter 3 gives applications of the Implicit Function Theorem. These are not bifurcation problems. Some of the applications were chosen because the material is needed in later chapters. Others give good illustrations of some of the tools in Chapter 2.

Chapters 4–8 deal with static bifurcation theory. Chapter 4 contains the fundamental elements of variational theory together with serious applications to Hamiltonian systems, elliptic and hyperbolic problems.

Chapters 5–8 deal almost entirely with analytic methods in local static bifurcation theory. In Chapter 5, for functions depending on a scalar parameter, conditions are given to ensure that there is always a bifurcation near equilibrium. These conditions are based on the linear approximation and are independent of the nonlinearities. Some global results are also included.

In Chapter 6, the case of a one-dimensional null space for the linear approximation is analyzed in detail under generic conditions on the quadratic and cubic terms. The effects of symmetry are also discussed. Chapter 7 is concerned with the case where the linear approximation has a two-dimensional null space with the quadratic and cubic terms satisfying some nondegeneracy conditions. Both of these chapters contain constructive procedures in the analysis. Chapter 8 contains applications to the buckling of plates, chemical reactions and Duffing's equation.

Chapters 9–13 are devoted to dynamic bifurcation theory. Chapter 9 is concerned with the bifurcation from an equilibrium point in the case when the linear approximation has either one zero eigenvalue or a pair of purely imaginary eigenvalues. It is shown that all relevant information on existence and stability is contained in the bifurcation function obtained via the alternative method or the method Liapunov–Schmidt. The hypotheses on the linear part are the typical situation for one parameter families of vector fields. Chapter 10 is devoted to the other bifurcation phenomena that occur in the plane for typical one parameter families of autonomous vector fields.

In Chapter 11, we discuss periodic planar vector fields and especially Hamiltonian systems with a small damping and small periodic forcing term. Emphasis is placed on the existence of subharmonic solutions and the role of successive bifurcations through subharmonics in the creation of homoclinic points and a type of random behavior.

In Chapter 12, averaging, the theory of normal forms and the theory of integral manifolds for ordinary differential equations are presented. This material is relevant to the discussion of bifurcation to tori considered in that chapter as well as the problems in Chapter 13, which is devoted to the behavior of the solutions of a differential equation near an equilibrium point when the linear part of the vector field is typical of two parameter problems.

The topics in Chapter 14 on perturbation of the spectrum of linear operators is distinct from the ones in the previous chapters. It is included because the same methods can be applied to yield elementary proofs of some results in this field.

The material in this book can be easily adapted to several types of one semester courses. For example, four possible reasonable arrangements could be:

I. Chapters 1, 5, 6, 7, 8 with Sections 2.3–2.8 from Chapter 2.
II. Chapter 2, 3, 4.
III. Chapters 1, 9, 10 with Sections 2.3, 2.4, 2.5 from Chapter 2.
IV. Chapters 11, 12, 13.

Examples I, II, III are independent and require minimal knowledge of differential equations. Example IV can only be taught after III and requires more sophisticated concepts from differential equations.

The authors are indebted to numerous colleagues and students for their assistance in this work. We especially thank John Mallet–Paret with whom we had so many stimulating conversations about technique and method of presentation. Luis Magalhães also was of great assistance, especially in the presentation in Chapter 11 and the examples in Chapter 12. We have also been assisted by many persons in the preparation of the final manuscript. We are indebted especially to Eleanor Addison, Dorothy Libutti, Sandra Spinacci, Kate MacDougall, Mary Reynolds and Diane Norton. The second author is also indebted to the Guggenheim Foundation for a Fellowship during 1979–80.

Contents

Chapter 1
Introduction and Examples ... 1
1.1. Definition of Bifurcation Surface ... 1
1.2. Examples with One Parameter ... 3
1.3. The Euler–Bernoulli Rod ... 5
1.4. The Hopf Bifurcation ... 9
1.5. Some Generic Examples ... 12
1.6. Dynamic Bifurcation ... 15

Chapter 2
Elements of Nonlinear Analysis ... 19
2.1. Calculus ... 19
2.2. Local Implicit Function Theorem ... 24
2.3. Global Implicit Function Theorem ... 27
2.4. Alternative Methods ... 30
2.5. Embedding Theorems ... 34
2.6. Weierstrass Preparation Theorem ... 36
2.7. The Malgrange Preparation Theorem ... 43
2.8. Newton Polygon ... 45
2.9. Manifolds and Transversality ... 51
2.10. Sard's Theorem ... 54
2.11. Topological Degree, Index of a Vector Field and Fixed Point Index ... 65
2.12. Ljusternik–Schnirelman Theory in $\mathbb{R}^n$ ... 78
2.13. Bibliographical Notes ... 85

Chapter 3
Applications of the Implicit Function Theorem ... 89
3.1. Existence of Solutions of Ordinary Differential Equations ... 89
3.2. Admissible Classes in Ordinary Differential Equations ... 90
3.3. Global Boundary Value Problems for Ordinary Differential Equations ... 92
3.4. Hopf Bifurcation Theorem ... 98
3.5. Liapunov Center Theorem ... 99

3.6. Saddle Point Property 102
3.7. The Hartman–Grobman Theorem 108
3.8. An Elliptic Problem 110
3.9. A Hyperbolic Problem 112
3.10. Bibliographical Notes 113

Chapter 4
Variational Method 115
4.1. Introduction 115
4.2. Weak Lower Semicontinuity 115
4.3. Monotone Operators 124
4.4. Condition (C) 129
4.5. Minimax Principle in Banach Spaces 134
4.6. Mountain Pass Theorem 140
4.7. Periodic Solutions of a Semilinear Wave Equation 142
4.8. Ljusternik–Schnirelman Theory on Banach Manifolds 147
4.9. Stationary Waves 151
4.10. The Krasnoselski Theorems 157
4.11. Variational Property of Bifurcation Equation 160
4.12. Liapunov Center Theorem at Resonance 162
4.13. Bibliographical Notes 166

Chapter 5
The Linear Approximation and Bifurcation 168
5.1. Introduction 168
5.2. Eigenvalues of B 169
5.3. Eigenvalues of (B, A) 174
5.4. Eigenvalues of $(B, A_1, \ldots, A_N)$ 177
5.5. Bifurcation from a Simple Eigenvalue 186
5.6. Applications of Simple Eigenvalues 191
5.7. Bifurcation Based on the Linear Equation 201
5.8. Global Bifurcation 206
5.9. An Application to a Delay Differential Equation 209
5.10. Bibliographical Notes 213

Chapter 6
Bifurcation with One Dimensional Null Space 215
6.1. Introduction 215
6.2. Quadratic Nonlinearities 217
6.3. Applications 222
6.4. Cubic Nonlinearities 226
6.5. Applications 230
6.6. Bifurcation from Known Solutions 233

6.7. Effects of Symmetry 236
6.8. Universal Unfoldings 241
6.9. Bibliographical Notes 242

Chapter 7
Bifurcation with Higher Dimensional Null Spaces 244
7.1. Introduction 244
7.2. The Quadratic Revisited 245
7.3. Quadratic Nonlinearities I 248
7.4. Quadratic Nonlinearities II 258
7.5. Cubic Nonlinearities I 263
7.6. Cubic Nonlinearities II 270
7.7. Cubic Nonlinearities III 274
7.8. Bibliographical Notes 279

Chapter 8
Some Applications 284
8.1. Introduction 284
8.2. The von Kármán Equations 284
8.3. The Linearized Problem 286
8.4. Noncritical Length 288
8.5. Critical Length 290
8.6. An Example in Chemical Reactions 294
8.7. The Duffing Equation with Harmonic Forcing 301
8.8. Bibliographical Notes 309

Chapter 9
Bifurcation near Equilibrium 311
9.1. Introduction 311
9.2. Center Manifolds 312
9.3. Autonomous Case 323
9.4. Periodic Case 329
9.5. Bifurcation from a Focus 334
9.6. Bibliographical Notes 343

Chapter 10
Bifurcation of Autonomous Planar Equations 349
10.1. Introduction 349
10.2. Periodic Orbit 349
10.3. Homoclinic Orbit 353

10.4. Closed Curve with a Saddle-Node 360
10.5. Remarks on Structural Stability and Bifurcation 362
10.6. Remarks on Infinite Dimensional Systems and Turbulence 365
10.7. Bibliographical Notes 367

Chapter 11
Bifurcation of Periodic Planar Equations 368
11.1. Introduction 368
11.2. Periodic Orbit-Subharmonics 369
11.3. Homoclinic Orbit 378
11.4. Subharmonics and Homoclinic Points 388
11.5. Abstract Bifurcation near a Closed Curve 395
11.6. Bibliographical Notes 397

Chapter 12
Normal Forms and Invariant Manifolds 401
12.1. Introduction 401
12.2. Transformation Theory and Normal Forms 402
12.3. More on Normal Forms 410
12.4. The Method of Averaging 420
12.5. Integral Manifolds and Invariant Tori 425
12.6. Bifurcation from a Periodic Orbit to a Torus 432
12.7. Bifurcation of Tori 439
12.8. Bibliographical Notes 441

Chapter 13
Higher Order Bifurcation near Equilibrium 443
13.1. Introduction 443
13.2. Two Zero Roots I 444
13.3. Two Zero Roots II 450
13.4. Two Zero Roots III 458
13.5. Several Pure Imaginary Eigenvalues 462
13.6. Bibliographical Notes 465

Chapter 14
Perturbation of Spectra of Linear Operators 467
14.1. Introduction 467
14.2. Continuity Properties of the Spectrum 468
14.3. Simple Eigenvalues 473

14.4. Multiple Normal Eigenvalues 481
14.5. Self-adjoint Operators 485
14.6. Bibliographical Notes 489

Bibliography 491

Index 513

Chapter 1

Introduction and Examples

1.1. Definition of Bifurcation Surface

Every problem in applications contains several physical parameters which may vary over certain specified sets. Thus, it is important to understand the qualitative behavior of the system as the parameters vary. A good design for a system will always be such that the qualitative behavior does not change when the parameters are varied a small amount about the value for which the original design was made. However, the behavior may change when the system is subjected to large variations in the parameters. A change in the qualitative properties could mean a change in stability of the original system and thus the system must assume a state different from the original design. In vague terms, the values of the parameters where this change takes place are called bifurcation values. Knowledge of the bifurcation values is absolutely necessary for the complete understanding of the system. Our objective is to give an overall view of methods and results in this area. To facilitate the introductory discussion, we first define the basic problem and give a precise definition of a bifurcation value.

Suppose X, Z are Banach spaces, Λ is an open set in a Banach space, $M: \Lambda \times X \to Z$ is continuous together with its first Frechet derivative. The set Λ will be called the *parameter set*. Sometimes, more derivatives on M are required and it will always be assumed that M has as many derivatives as necessary if it is not always explicitly stated. Consider the equation

$$M(\lambda, x) = 0 \tag{1.1}$$

for $\lambda \in \Lambda$, $x \in X$.

A *solution* of Equation (1.1) is a point $(\lambda, x) \in \Lambda \times X$ such that Equation (1.1) is satisfied. Let $S \subset \Lambda \times X$ denote the set of solutions of Equation (1.1) and, for any $\lambda \in \Lambda$, let

$$S_\lambda = \{x \in X : (\lambda, x) \in S\}.$$

In a physical system, Equation (1.1) generally represents the equilibrium positions of the system or, more generally, equations for the state of the system which satisfy certain boundary conditions. The dynamics of the system are not included in Equation (1.1). Stability of a solution of Equation (1.1) often requires a discussion of a differential equation $du/dt = \bar{M}(u, \lambda)$ for u near $x \in S_\lambda$ and $\bar{M}$ related to M in some way.

If S is any closed set in $\Lambda \times X$, one can always construct an M such that S is the solution set for Equation (1.1). As a consequence, it is impossible to give a complete description of S_λ as λ varies. In the applications, the most typical solution sets have components consisting of various pieces of the solution set depicted in Figure 1.1.

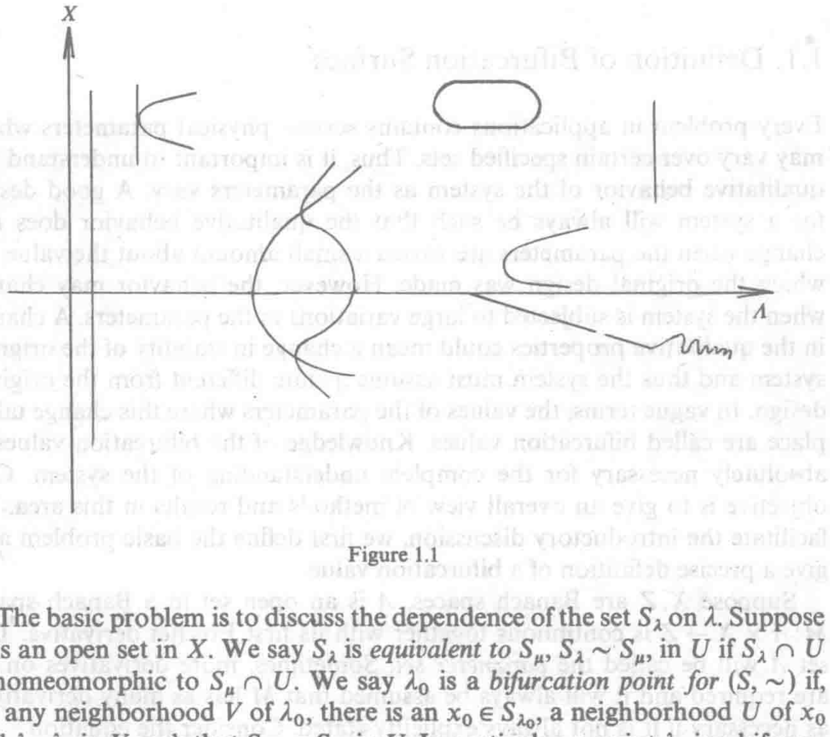

Figure 1.1

The basic problem is to discuss the dependence of the set S_λ on λ. Suppose U is an open set in X. We say S_λ is *equivalent to* S_μ, $S_\lambda \sim S_\mu$, in U if $S_\lambda \cap U$ is homeomorphic to $S_\mu \cap U$. We say λ_0 is a *bifurcation point for* $(S, \sim)$ if, for any neighborhood V of λ_0, there is an $x_0 \in S_{\lambda_0}$, a neighborhood U of x_0 and λ_1, λ_2 in V such that $S_{\lambda_1} \nsim S_{\lambda_2}$ in U. In particular, a point λ_0 is a bifurcation point if $S_{\lambda_0} \neq \emptyset$, the empty set, and there is an $x_0 \in S_{\lambda_0}$ such that, for any neighborhood U of (λ_0, x_0), there are two distinct solutions (λ, x_1), $(\lambda, x_2) \in U$; that is, there is a $\lambda \in \Lambda$, $x_1, x_2 \in X$, $x_1 \neq x_2$, such that (λ, x_1), $(\lambda, x_2) \in U$ and (λ, x_1), (λ, x_2) satisfy Equation (1.1). Whenever possible, it is desirable to have the complete characterization of the solution set S in a neighborhood U of a solution (λ_0, x_0) for which λ_0 is a bifurcation point.

The Implicit Function Theorem shows that the characterization is trivial near some solutions (λ_0, x_0).

Lemma 1.1. *If $(\lambda_0, x_0) \in S$ and $D_x M(\lambda_0, x_0)$ has a bounded inverse, then there is a neighborhood Ω of (λ_0, x_0) such that $S \cap \Omega$ is a diffeomorphic image of a* *neighborhood of λ_0; more precisely, there is a neighborhood Λ_0 of λ_0 and a continuously differentiable function $x^*: \Lambda_0 \to X$ such that $S \cap \Omega = \{(\lambda, x^*(\lambda)), \lambda \in \Lambda_0\}$.*

As a consequence of Lemma 1.1, the only points $(\lambda_0, x_0) \in S$ that require further discussion are those for which $D_x M(\lambda_0, x_0)$ is singular.

1.2. Examples with One Parameter

If $X = Z$, $M(\lambda, x) = Bx - \lambda x$, where $\lambda \in \mathbb{R}$ and $B: X \to X$ is a bounded linear operator, then any eigenvalue λ_0 of B is a bifurcation point. In fact, if λ_0 is an eigenvalue of B, then, for any $\varepsilon > 0$, there is an $x_0 \in X$, $|x_0| = \varepsilon$ such that $Bx_0 = \lambda_0 x_0$. On the other hand, if

$$M(\lambda, x) = Bx - \lambda x + 0(|x|^2 + |\lambda - \lambda_0|^2|x|) \tag{2.1}$$

as $x \to 0$, $\lambda \to \lambda_0$ where λ_0 is an eigenvalue of B, the point λ_0 may not be a bifurcation point. Notice that $M(\lambda, 0) = 0$ for all $\lambda \in \Lambda$ so that $(\lambda, 0)$ is a solution. In fact, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the equations

$$\begin{aligned} x_2(x_1^2 + x_2^2) + \lambda x_1 &= 0 \\ -x_1(x_1^2 + x_2^2) + \lambda x_2 &= 0. \end{aligned} \tag{2.2}$$

Any solution of Equation (2.2) must satisfy $(x_1^2 + x_2^2)^2 = 0$ for all λ; that is, $x_1 = x_2 = 0$. Thus, $\lambda = 0$ is not a bifurcation point and it is a double eigenvalue of the linear part of Equation (2.2).

To emphasize the role of the nonlinearities in Equation (2.2), we make only a sign change and bifurcation will occur at $\lambda = 0$. In fact, the equations

$$\begin{aligned} x_2(x_1^2 + x_2^2) + \lambda x_1 &= 0 \\ x_1(x_1^2 + x_2^2) + \lambda x_2 &= 0 \end{aligned} \tag{2.3}$$

have the solutions $x_2 = x_1$, $x_1(2x_1^2 + \lambda) = 0$. Thus, $\lambda = 0$ is a bifurcation point.

It is also possible to give a similar example in $\mathbb{R}^2$ for the case in which the null space $\mathcal{N}(B)$ has dimension one. In fact, consider the equation

$$\begin{aligned} x_2 + \lambda x_1 &= 0 \\ \lambda x_2 - x_1^3 &= 0 \end{aligned} \tag{2.4}$$

near $(\lambda_0, x_{10}, x_{20}) = (0, 0, 0)$. The matrix B for this case is

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has $\lambda_0 = 0$ as an eigenvalue of multiplicity two but $\mathcal{N}(B)$ has dimension one. The above equation is equivalent to $x_2 + \lambda x_1 = 0, x_1(\lambda^2 + x_1^2) = 0$ which has only the solution $x_1 = x_2 = 0$ for all $\lambda \in \mathbb{R}$. Therefore, $\lambda_0 = 0$ is not a bifurcation point.

If $\lambda_0 \in \mathbb{R}$ is a simple eigenvalue of B and M is given as in Relation (2.1), then λ_0 is always a bifurcation point. Although an even more general result will be given in Chapter 4, it is instructive to do the simplest case here. Suppose $X = Z = \mathbb{R}^n$, $\Lambda = \mathbb{R}$ and M is given in Relation (2.1). If λ_0 is a simple eigenvalue of B, then we may assume that $B = \text{diag}(0, B_0)$ where $B_0 - \lambda_0 I$ is an $(n-1)x(n-1)$ nonsingular matrix. If $x = (y, z)$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$, then $M(\lambda, x) = 0$ is equivalent to:

$$(2.5)\quad \begin{aligned} &\text{(a)} \qquad \mu y = g(\mu, y, z) \\ &\text{(b)} \qquad (B_0 - \lambda_0 I)z = h(\mu, y, z) \end{aligned}$$

where $\mu = \lambda - \lambda_0$, and both g, h are $0(y^2 + |z|^2 + \mu^2(|y| + |z|))$ as $y, z, \mu \to 0$. Since $B_0 - \lambda_0 I$ is nonsingular, one can invoke the Implicit Function Theorem to solve Equation (2.5b) for $z = z^*(\mu, y)$ near $\mu = 0$, $y = 0$, $z^*(\mu, 0) = 0$. Therefore, Equations (2.5) are equivalent to the scalar equation

$$(2.6)\qquad \mu y = g(\mu, y, z^*(\mu, y)).$$

Since $z^*(\mu, 0) = 0$, $g(\mu, 0, 0) = 0$, one can divide Equation (2.6) by y to obtain the equivalent equation for $y \neq 0$

$$(2.7)\qquad \mu = \bar{g}(\mu, y)$$

where $\bar{g}(\mu, y) = g(\mu, y, z^*(\mu, y))/y$ satisfies $\bar{g}(0,0) = 0$, $D_\mu \bar{g}(0,0) \neq 0$. Thus, the Implicit Function Theorem implies there is a solution $\mu = \mu^*(y)$ for y near zero and $\mu^*(0) = 0$. This proves $\mu = 0$ (or $\lambda = \lambda_0$) is a bifurcation point. In Figure 2.1, we have depicted a possible solution set. Without further restrictions on the nonlinearity, we can only assert that the curve $\mu = \mu^*(y)$ is a smooth curve passing through $(0, 0)$.

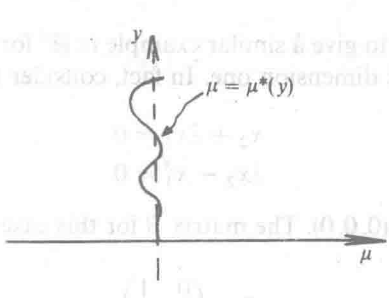

Figure 2.1

The procedure used in the above example is known as the method of Liapunov-Schmidt or the method of alternative problems. The method in abstract form is fundamental in bifurcation theory.

1.3. The Euler–Bernoulli Rod

To illustrate some other ideas that occur in bifurcation theory, let us consider the Euler–Bernoulli problem of the buckling of a rod. In equilibrium position, suppose the rod of length l is represented in the (x, y)-plane by the set $\{(x,0), 0 \leq x \leq l\}$, the rod is fixed at $(0,0)$ and the other end $(l,0)$ is allowed to vary along the x-axis when it is subjected to a constant horizontal force P at the right end. If s represents arc length along the displaced rod, $\phi(s)$ represents the angle which the unit tangent vector to the rod makes with the x-axis (see Figure 3.1) and it is assumed that the change in curvature is proportional to the moment of force, then the equations describing the displacement of the rod are

$$(3.1)\qquad Py = -k\frac{d\phi}{ds}, \qquad \frac{dy}{ds} = \sin\phi$$

where k is a constant and the boundary conditions are

$$(3.2)\qquad y(0) = y(l) = 0, \qquad x(0) = 0.$$

If $P \neq 0$, Equations (3.1) and Boundary Conditions (3.2) are equivalent to

$$(3.3)\qquad \frac{d^2\phi}{ds^2} + \lambda \sin\phi = 0, \qquad \phi'(0) = \phi'(l) = 0.$$

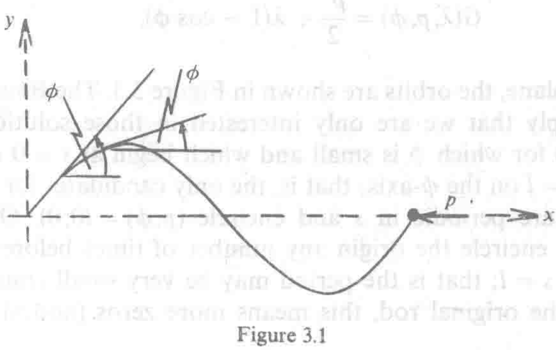

Figure 3.1

Let $C^k[0,l] = \{f:[0,l] \to \mathbb{R}: f$ is continuous together with derivatives up through order $k\}$. For any $f \in C^k[0,l]$, let

$$|f|_k = \max\{\sup[|f^{(j)}(s)|:0 \le s \le l], j = 0,1,\ldots,k\}.$$

Let $X = \{f \in C^2[0,l]: f'(0) = f'(l) = 0\}$, $Z = C^0[0,l]$ and define

$$F:\mathbb{R} \times X \to Z, \qquad F(\lambda,\phi) = \frac{d^2\phi}{ds^2} + \lambda \sin\phi. \tag{3.4}$$

Our objective is to determine all solutions of the equation

$$F(\lambda,\phi) = 0, \qquad (\lambda,\phi) \in \mathbb{R} \times X \tag{3.5}$$

which have ϕ "close" to the trivial solution $\phi = 0$.

Let us first consider the linear problem

$$D_\phi F(\lambda,0)\psi = 0 \tag{3.6}$$

which is equivalent to

$$\begin{aligned} &\frac{d^2\psi}{ds^2} + \lambda\psi = 0 \\ &\psi'(0) = \psi'(l) = 0. \end{aligned} \tag{3.7}$$

It is easy to verify that this equation has a nontrivial solution if and only if $\lambda = \lambda_m = m^2\pi^2/l^2$, $m = 0,1,2,\ldots$ and, for $\lambda = \lambda_m$, every solution is a constant multiple of $\psi_m(s) = \cos\sqrt{\lambda_m}s$. For the original problem of the bar, $m = 0$ is not possible since $m = 0$ and $P \neq 0$ implies that $y(s) = 0$, $0 \le s \le l$. Figure 3.2 shows the displaced rod corresponding to λ_m for $m = 1,2,3$.

If $\phi(s)$ is a solution of Equation (3.3) and $p(s) = d\phi(s)/ds$, then

$$\begin{aligned} &G(\lambda,p(s),\phi(s)) = G(\lambda,p(0),\phi(0)), \qquad 0 \le s \le l, \\ &G(\lambda,p,\phi) = \frac{p^2}{2} + \lambda(1-\cos\phi). \end{aligned} \tag{3.8}$$

In the (p,ϕ)-plane, the orbits are shown in Figure 3.3. The Boundary Condition (3.2) imply that we are only interested in those solutions satisfying Relation (3.8) for which ϕ is small and which begin at $s = 0$ on the ϕ-axis and end at $s = l$ on the ϕ-axis; that is, the only candidates for solutions are those which are periodic in s and encircle $(p,\phi) = (0,0)$. Of course, the solution may encircle the origin any number of times before it returns to the ϕ-axis at $s = l$; that is the period may be very small compared with l. In terms of the original rod, this means more zeros (nodes) as shown in Figure 3.2.