PLANE AND SPHERICAL TRIGONOMETRY

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PREFACE

This book is designed for those who wish to master the fundamental principles of Trigonometry and its most important applications. It is adapted to the use of colleges and high schools.

The proofs of formulas are simple but rigorous. The use of directed lines is consistent; the directions of such lines in the figures are usually indicated by arrowheads, and these lines are always read from origin to end. Both trigonometric ratios and trigonometric lines are employed, but at first the ratios are used exclusively until they have become fixed in the mind and have been made familiar by use in the solution of right triangles.

The distinction between identities and equations is recognized in definition, notation, and treatment. The solution of trigonometric equations is scientific and complete. The trigonometric ratios are defined in pairs as reciprocals of each other both to aid the memory and to emphasize one of the most important of their fundamental relations. The addition formulas are proved for positive or negative angles of any quadrant, and from them are deduced the other formulas concerning the functions of two or more angles. When two or more figures are used in a proof, the same phraseology always applies to each figure.

In the first chapter, by means of the right triangle, the pupil is taught some of the uses of Trigonometry before he is required to master the broader ideas and relations of Analytic

Trigonometry; but at the same time the emphasis is so distributed that when the general ideas are taken up they easily replace the special ones.

In Chapter VIII complex number is expressed as an arithmetic multiple of a quality unit in its trigonometric type form, and the fundamental properties of such number are demonstrated. The proof of De Moivre's theorem is simple but complete, and its meaning and uses are illustrated by examples.

In Spherical Trigonometry the fundamental relations of spherical angles and triangles to diedral and triedral angles are illustrated by constructions. The complete solution of the right triangle is discussed by itself, but later the formulas used are shown to be only special cases of the laws of sines and cosines for the oblique triangle. The most useful and interesting problems have been selected and special attention has been given to methods of solution and to arrangement of work.

It is believed that the order of the text is the best for beginners; but, with the exception of a few articles, Chapter I or Chapter X may be omitted by those who are prepared to take up at once the general treatment in Chapter II or Chapter XI. Too much stress cannot be laid on careful and accurate construction and measurement in the first chapters. Chapters VII and VIII and the latter part of Chapter VI may be omitted by those who wish a shorter course.

In writing this book the author has consulted the best authorities, both American and European. Many of the examples have been taken from these sources. The author takes this opportunity to express to many teachers and other friends his appreciation of their valuable suggestions in the course of the preparation of the book.

JAMES M. TAYLOR

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PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC RATIOS OF ACUTE ANGLES

1. Let A denote the *number* of degrees in the acute angle XOB; then A is the *numerical measure*, or *measure*, of this angle, and we can write $\angle XOB = A$.

From any point in either side of the angle XOB, as P, draw PM perpendicular to the other side.

Observe that O is the vertex of the angle, and M is the foot of the perpendicular drawn from P. This lettering should be fixed in mind so that in the following definitions the lines MP, OM, and OP shall always mean the same lines as in fig. 1.

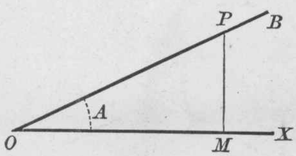

FIG. 1

The six simple ratios (three ratios and their reciprocals) which can be formed with the three lines MP, OM, OP are called the *trigonometric ratios* of the angle XOB, or A.

These ratios are named as follows:

The *ratio*	**MP / OP**	is the **sine** of A;
and its *reciprocal*	**OP / MP**	is the **cosecant** of A.
The *ratio*	**OM / OP**	is the **cosine** of A;
and its *reciprocal*	**OP / OM**	is the **secant** of A.
The *ratio*	**MP / OM**	is the **tangent** of A;
and its *reciprocal*	**OM / MP**	is the **cotangent** of A.

For brevity the *sine* of A is written **sin A**; the *cosine* of A, **cos A**; the *tangent* of A, **tan A**; the *cotangent* of A, **cot A**; the *secant* of A, **sec A**; and the *cosecant* of A, **csc A**.

Observe that *sin A* is a compound symbol which, taken as a whole, denotes a *number*. The same is true of cos A, tan A, etc.

Ex. 1. What four trigonometric ratios of the angle A involve the line MP? the line OM? the line OP?

Ex. 2. What trigonometric ratios are reciprocals of each other?

Ex. 3. Which is the greater, tan A or sec A? cot A or csc A? Why?

Ex. 4. Can sin A or cos A exceed 1? Why?

2. *Any trigonometric ratio of a given angle has only one value.*

Let XOB be any acute angle. Draw $PM \perp OX$, $P'M' \perp OX$, $P''M'' \perp OB$; then, by § 1,

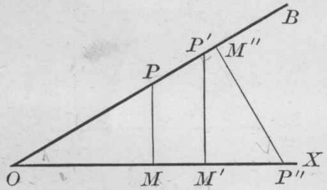

FIG. 2

$$\sin XOB = \frac{MP}{OP},\ \frac{M'P'}{OP'},$$

or $$M''P''/OP''. \qquad (1)$$

From the similarity of the △ OMP, $OM'P'$, $OM''P''$ it follows that the three ratios in (1) (or their reciprocals) are all equal; hence sin XOB (or csc XOB) has but one value.

Also, from the similarity of these △, each of the other trigonometric ratios of $\angle XOB$ has only one value.

3. *Two* **acute** *angles are equal if any trigonometric ratio of the one is equal to the same ratio of the other.*

Take O_1P_1 in fig. 3 equal to OP in fig. 2, and draw $P_1M_1 \perp O_1X_1$. We are to prove that

if $$\sin X_1O_1P_1 = \sin XOP,$$

i.e. if $$M_1P_1/O_1P_1 = MP/OP, \qquad (1)$$

then $$\angle X_1O_1P_1 = \angle XOP. \qquad (2)$$

By construction, $O_1P_1 = OP$.

Hence, from (1), $M_1P_1 = MP$.

Therefore, by Geometry, the right triangles $O_1P_1M_1$ and OPM are equal in all their parts.

Hence $\angle X_1O_1P_1 = \angle XOP$.

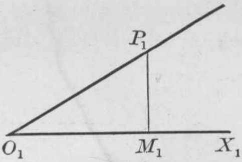

FIG. 3

In like manner the student should prove the equality of two acute angles when any other trigonometric ratio of the one is equal to the same ratio of the other.

4. *Having given the value of any trigonometric ratio of an acute angle, to construct the angle and obtain the values of its other trigonometric ratios.*

This problem will be illustrated by particular examples.

Ex. 1. If A is an acute angle and $\cos A = 3/5$, construct A and find the values of its other trigonometric ratios.

Here $\cos A = OM/OP = 3/5$.

Hence, if $OP = 5$ units, $OM = 3$ units.

Let O, in fig. 4, be the vertex of the angle A, and OX one of its sides.

On OX, to some scale, lay off OM equal to 3 units, and at M draw $MS \perp OX$.

With O as a center and with a radius equal to 5 units, draw an arc cutting MS in some point as P. Draw OPB.

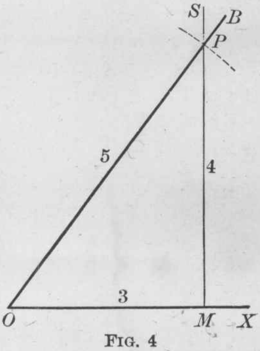

FIG. 4

Then $\angle XOB = A$.

For $\cos XOB = 3/5 = \cos A$.

Hence, by § 3, $\angle XOB = A$.

Again, $MP = \sqrt{5^2 - 3^2}$ units $= 4$ units.

Hence $\sin A = 4/5$, $\csc A = 5/4$; § 1

$\cos A = 3/5$, $\sec A = 5/3$;

$\tan A = 4/3$, $\cot A = 3/4$.

Observe that 5 is the *numerical measure* of OP, 4 of MP, and 3 of OM.

Ex. 2. If A is an acute angle and $\sin A = 2/3$, construct A and find the values of its other trigonometric ratios.

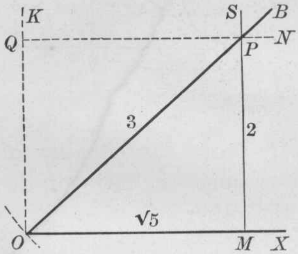

FIG. 5

Here $\sin A = MP/OP = 2/3$.

Hence, if $OP = 3$ units,

$MP = 2$ units.

At M, in fig. 5, draw $MS \perp OX$ and lay off MP equal to 2 units.

With P as a center and 3 units as a radius, strike an arc cutting OX in some point, as O. Draw OPB.

Then $\angle XOB = A$.

For $\sin XOB = 2/3 = \sin A$.

Hence, by § 3, $\angle XOB = A$.

Again, $OM = \sqrt{3^2 - 2^2}$ units $= \sqrt{5}$ units.

Hence $\sin A = 2/3$, $\csc A = 3/2$; § 3

$\cos A = \sqrt{5}/3$, $\sec A = 3/\sqrt{5}$;

$\tan A = 2/\sqrt{5}$, $\cot A = \sqrt{5}/2$.

If the vertex of the angle A were required to be at some fixed point on OX, as O, we would draw $OK \perp OX$, lay off OQ equal to 2 units, through Q draw QN parallel to OX, and with O as a center and 3 units as a radius, strike an arc cutting QN at P; then draw OPB as before.

By using a protractor, *i.e.* a graduated semicircle, we find that $\angle XOB$, or A, is an angle of about 42°.

NOTE. The student should form the habit of gaining a clear idea of the size of an angle from the value of any one of its trigonometric ratios.

EXERCISE I

Construct the acute angle A, obtain the values of all its trigonometric ratios, and find its size in degrees, when:

1. $\sin A = 2/5$.	**6.** $\tan A = 4/3$.	**11.** $\csc A = 5/2$.
2. $\sin A = 4/5$.	**7.** $\cot A = 5/2$.	**12.** $\csc A = 3/2$.
3. $\cos A = 3/4$.	**8.** $\cot A = 1/3$.	**13.** $\tan A = 4$, or $4/1$.
4. $\cos A = 1/3$.	**9.** $\sec A = 5/3$.	**14.** $\cot A = 7$, or $7/1$.
5. $\tan A = 1/4$.	**10.** $\sec A = 4/3$.	**15.** $\tan A = 9$.

16. Express each of the trigonometric ratios of an acute angle A in terms of its sine, writing $(\sin A)^2$ in the form $\sin^2 A$.

In fig. 1, let $OP = 1$.

Then $MP/OP = MP/1$, *i.e.* $\sin A$ is the *measure* of MP.

Whence $MP = \sin A$,

and $OM = \sqrt{\overline{OP}^2 - \overline{MP}^2} = \sqrt{1 - \sin^2 A}$.

Hence $\cos A = OM/OP = \sqrt{1 - \sin^2 A}$;

$\therefore \sec A = 1/\sqrt{1 - \sin^2 A}$. § 1

$\tan A = MP/OM = \sin A/\sqrt{1 - \sin^2 A}$;

$\therefore \cot A = \sqrt{1 - \sin^2 A}/\sin A$.

$\csc A = OP/MP = 1/\sin A$.

17. Express each of the trigonometric ratios of an acute angle in terms of its cosine.

In fig. 1, let $OP = 1$.

Then $OM/OP = OM/1$, *i.e.* $\cos A$ is the *measure* of OM.

Whence $OM = \cos A$,

and $MP = \sqrt{\overline{OP}^2 - \overline{OM}^2} = \sqrt{1 - \cos^2 A}$, etc.

18. Express each of the trigonometric ratios of an acute angle in terms of its tangent.

In fig. 1, let $OM = 1$.

Then $MP/OM = MP/1$, *i.e.* $\tan A$ is the *measure* of MP.

Whence $MP = \tan A$,

and $OP = \sqrt{\overline{OM}^2 + \overline{MP}^2} = \sqrt{1 + \tan^2 A}$, etc.

5. *To find approximately by measurement the values of the trigonometric ratios of any given angle.*

Ex. 1. Find by construction and measurement the values of the six trigonometric ratios of 40°.

With a protractor lay off $\angle XOB = 40°$.

Take OP any convenient length, say 10 units (the longer the better), and draw $PM \perp OX$. By careful measurement we find that

$MP = 6.4$ units,

$OM = 7.7$ units, approximately.

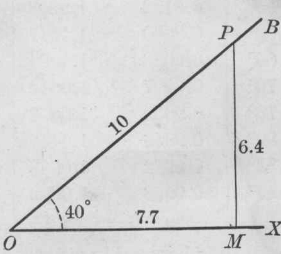

FIG. 6