Operator Theory
Advances and Applications
248

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Integral Operators in Non-Standard Function Spaces

Volume 1: Variable Exponent Lebesgue and Amalgam Spaces

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Preface

This book is a result of our ten-year fruitful collaboration. It deals with integral operators of harmonic analysis and their various applications in new, non-standard function spaces. Specifically, we deal with variable exponent Lebesgue and amalgam spaces, variable exponent Hölder spaces, variable exponent Campanato, Morrey and Herz spaces, Iwaniec–Sbordone (grand Lebesgue) spaces, grand variable exponent Lebesgue spaces, which unify two types of spaces mentioned above, grand Morrey spaces, generalized grand Morrey spaces, as well as weighted analogues of most of them.

In recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It thus became necessary to introduce and study the spaces mentioned above from various viewpoints. One of such spaces is the variable exponent Lebesgue space. For the first time this space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. In the beginning these spaces had theoretical interest. Later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$-Laplacian, in Zhikov [375, 377, 376, 379, 378], which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by appeared applications to various problems of Applied Mathematics, e.g., in modelling electrorheological fluids Acerbi and Mingione [3], Rajagopal and Růžička [301], Růžička [306] and more recently, in image restoration Aboulaich, Meskine, and Souissi [1], Chen, Levine, and Rao [42], Harjulehto, Hästö, Latvala, and Toivanen [127], Rajagopal and Růžička [301].

Variable Lebesgue space appeared as a special case of the Musielak–Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz.

The large number of various results for non-standard spaces obtained during last decade naturally led us to two-volume edition of our book. In this Preface to Volume 1 we briefly characterize the book as a whole, and provide more details on the material of Volume 1.

Recently two excellent books were published on variable exponent Lebesgue spaces, namely:

> L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011,

and

> D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Foundations and Harmonic Analysis, Birkhäuser, Springer, Basel, 2013.

A considerable part of the first book is devoted to applications to partial differential equations (PDEs) and fluid dynamics. In the recent book

> V. Kokilashvili and V. Paatashvili, *Boundary Value Problems for Analytic and Harmonic Functions in Non-standard Banach Function Spaces*, Nova Science Publishers, New York, 2012,

there are presented applications to other fields, namely to boundary value problems, including the Dirichlet, Riemann, Riemann–Hilbert and Riemann–Hilbert–Poincaré problems. These problems are solved in domains with non-smooth boundaries in the framework of weighted variable exponent Lebesgue spaces.

The basic arising question is: what is the difference between this book and the above-mentioned books? What new theories and/or aspects are presented here? What is the motivation for a certain part of the book to treat variable exponent Lebesgue spaces? Below we try to answer these questions.

First of all, we claim that most of the results presented in our book deal with the integral transforms defined on general structures, namely, on measure metric (quasi-metric) spaces. A characteristic feature of the book is that most of statements proved here have the form of criteria (necessary and sufficient conditions).

In the part related to the variable exponent Lebesgue spaces in Volume 1 we single out the results for: weighted inequality criteria for Hardy-type and Carleman–Knopp operators, a weight characterization of trace inequalities for Riemann–Liouville transforms of variable order, two-weight estimates, and a solution of the trace problem for strong fractional maximal functions of variable order and double Hardy transforms. It should be pointed out that in this problem the situation is completely different when the fractional order is constant. Here two-weight estimates are derived without imposing the logarithmic condition for the exponents of spaces. We also treat boundedness/compactness criteria for weighted kernel operators including, for example, weighted variable-order fractional integrals.

For the variable exponent amalgam spaces we give a complete description of those weights for which the corresponding weighted kernel operators are bounded/compact. The latter result is new even for constant exponent amalgam spaces. We give also weighted criteria for the boundedness of maximal and potential operators in variable exponent amalgam spaces.

In Volume 1 we also present the results on mapping properties of one-sided maximal functions, singular, and fractional integrals in variable exponent Lebesgue spaces. This extension to the variable exponent setting is not only natural, but also has the advantage that it shows that one-sided operators may be bounded under weaker conditions on the exponent those known for two-sided operators. Among others, two-weight criterion is obtained for the trace inequality for one-sided potentials.

In this volume we state and prove results concerning mapping properties of hypersingular integral operators of order less than one in Sobolev variable exponent spaces defined on quasi-metric measure spaces. High-order hypersingular integrals are explored as well and applied to the complete characterization of the range of Riesz potentials defined on variable exponent Lebesgue spaces.

Special attention is paid to the variable exponent Hölder spaces, not treated in existing books. In the general setting of quasi-metric measure spaces we present results on mapping properties of fractional integrals whose variable order may vanish on a set of measure zero. In the Euclidean case our results hold for domains with no restriction on the geometry of their boundary.

The established boundedness criterion for the Cauchy singular integral operator in weighted variable exponent Lebesgue spaces is essentially applied to the study of Fredholm type solvability of singular integral equations and to the PDO theory. Here a description of the Fredholm theory for singular integral equations on composite Carleson curves oscillating near modes, is given using Mellin PDO.

In Volume 2 the mapping properties of basic integral operators of Harmonic Analysis are studied in generalized variable exponent Morrey spaces, weighted grand Lebesgue spaces, and generalized grand Morrey spaces. The grand Lebesgue spaces are introduced on sets of infinite measure and in these spaces boundedness theorems for sublinear operators are established. We introduce new function spaces unifying the variable exponent Lebesgue spaces and grand Lebesgue spaces. Boundedness theorems for maximal functions, singular integrals, and potentials in grand variable exponent Lebesgue spaces defined on spaces of homogeneous type are established.

In Volume 2 the grand Bochner–Lebesgue spaces are introduced and some of their properties are treated.

The entire book is mostly written in the consecutive way of presentation of the material, but in some chapters, for reader's convenience, we recall definitions of some basic notions. Although we use a unified notation in most of the cases, in some of the cases the notation in a chapter is specific for that concrete chapter.

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