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Dynamic Meteorology.

By

A. ELIASSEN and E. KLEINSCHMIDT jr.¹

With 41 Figures.

I. Basic theory.

a) Physical properties of atmospheric air.

1. **Fluid models.** All motions in the atmosphere connected with weather are formed as a result of heating from the sun. The atmosphere acts as a kind of heat engine, continually converting heat into mechanical energy; and mechanical energy is in turn converted back to heat by frictional dissipation. This indicates that neither pure hydrodynamics, nor pure thermodynamics suffices to explain the motions in the atmosphere; the appropriate science is a combination of both, which is called "physical hydrodynamics".

In order that a fluid shall be able to serve as the working substance of a heat engine, which converts heat into mechanical energy by circular processes without any progressive change of state of the fluid, it is necessary that the possible thermodynamic states of each fluid particle form a two-dimensional manifold, at least. This is the case for atmospheric air, where two functions of state, e.g. pressure and density, are free to vary independently under the influence of heat sources.

The theory of the motion of such a general fluid is rather complicated, and very few results have been obtained except some general theorems. Most theories in dynamic meteorology have been based on a simpler fluid model where the possible thermodynamic states of each fluid particle form a one-dimensional manifold. Such fluids were named *piezotropic* by V. BJERKNES and coll. [2].

A piezotropic fluid cannot convert heat into mechanical energy by circular processes, and the assumption that the air is piezotropic is therefore inadequate to explain the maintenance of atmospheric motions against frictional dissipation. Nevertheless, the concept of a piezotropic fluid has proven useful in the study of motion phenomena of relatively short time scale which utilize energy stored as potential, internal or kinetic energy. The hydrodynamics of piezotropic fluids has therefore played an important role in dynamic meteorology. In a piezotropic fluid, there is for each particle a relation between density ρ and pressure p . In the most general case, this *equation of piezotropy* may be different for all particles; it may then be written

$$\rho = f(p, A, B, C), \quad (1.1)$$

where A , B and C are a set of LAGRANGIAN coordinates, which define individual particles. Letting the symbol D denote the differential derived from the *individual* change of a quantity, i.e. the change experienced by a fluid particle, one has for the LAGRANGIAN coordinates

$$\frac{DA}{Dt} = \frac{DB}{Dt} = \frac{DC}{Dt} = 0. \quad (1.2)$$

¹ Chap. I to III are written mainly by ELIASSEN, Chap. IV and V by KLEINSCHMIDT.
Handbuch der Physik, Bd. XLVIII.

The differential quotient

$$\gamma = \frac{D\rho}{Dp} = \frac{\partial f}{\partial p} \quad (1.3)$$

is a measure of the compressibility and is called the *coefficient of piezotropy*.

It is not necessary that as many as three LANGRANGIAN coordinates appear in the equation of piezotropy. In the case most commonly considered, the adiabatic motion, this equation contains just one such parameter, viz. the entropy s . The equation of piezotropy then assumes the form

$$\rho = f(p, s) \quad (1.4)$$

with the condition

$$Ds/Dt = 0. \quad (1.5)$$

In this case, the relationship between pressure and density will be the same for particles with the same entropy, but will be different for particles with different entropy.

Another example of piezotropy is the *incompressible* fluid (provided its density is independent of the temperature), for which pressure drops out in the equation of piezotropy.

An important special case occurs if the relationship between pressure and density is the same for all particles within the fluid. In this case, no parameters appear in the equation of piezotropy, so that this equation becomes

$$\rho = f(p). \quad (1.6)$$

The surfaces of constant density must then at all times coincide with the surfaces of constant pressure (barotropy), and the fluid is said to be *autobarotropic*. In an autobarotropic fluid the interchange of two particles will not cause any change of the field of any function of state. An adiabatic fluid in which all particles have the same entropy, and a homogeneous and incompressible fluid are both examples of autobarotropic fluids.

The "classical" hydrodynamics may be defined as the hydrodynamics of autobarotropic fluids, in particular of the homogeneous and incompressible fluid. In contrast, the "physical" hydrodynamics is the hydrodynamics of piezotropic fluids and more general fluid models of two or more independent variables of state.

The theoretical basis of physical hydrodynamics was laid by V. BJERKNES, who systematized this science, and introduced the classification of fluid models outlined above, and by MARGULES, who studied the energy conversions in general fluid models.

2. Thermodynamic properties of dry air. Atmospheric air is a mixture of a number of perfect gases and water vapor, sometimes also containing suspended condensation products of water. Whereas the amount of water vapor is highly variable, the mixing ratio of the perfect gases is constant up to great heights. Dry air is therefore a well-defined gaseous mixture which with great accuracy behaves like an ideal gas.

Thus the pressure p , density ρ and absolute temperature T of dry air fulfill the gas equation

$$p/\rho = RT, \quad (2.1)$$

where $R = 2.8704 \times 10^6 \text{ cm}^2 \text{ sec}^{-2} \text{ per degree absolute}$, is the gas constant of dry air.

The *internal energy* per unit mass of dry air is

$$e = c_v T + \text{constant}, \quad (2.2)$$

where $c_v = 7.17 \times 10^6 \text{ cm}^2 \text{ sec}^{-2} \text{ per degree}$ is the specific heat at constant volume.

The *enthalpy* per unit mass is

$$h = e + p\alpha = c_p T + \text{constant}, \quad (2.3)$$

where $\alpha = \rho^{-1}$ is the specific volume, and

$$c_p = c_v + R. \quad (2.4)$$

The ratio

$$\kappa = \frac{c_p}{c_v} \quad (2.5)$$

has for dry air the value 1.4.

For any gas or liquid, or for any gaseous mixture of constant mixing ratio, the *entropy* per unit mass is defined by

$$T ds = de + p d\alpha = dh - \alpha dp. \quad (2.6)$$

It should be noted that the differentials in this equation are quite general; the equation holds for individual changes as well as for changes from one particle to another.

In the case of dry air, one has

$$T ds = c_v dT + p d\alpha = c_p dT - \alpha dp. \quad (2.7)$$

Eliminating α by means of (2.1) and integrating, one obtains

$$s = c_p \log T - R \log p + \text{constant} = c_v \log p - c_p \log \rho + \text{constant}. \quad (2.8)$$

A convenient measure of the entropy of dry air is the *potential temperature*, θ , which may be defined by

$$s = c_p \log \theta + \text{constant}. \quad (2.9)$$

Comparison between (2.8) and (2.9) shows that

$$\theta = \left(\frac{p_0}{p}\right)^{1-\frac{1}{\kappa}} T = \frac{p_0^{1-\frac{1}{\kappa}} p^{\frac{1}{\kappa}}}{R\rho}, \quad (2.10)$$

where p_0 is a constant which is given the value 1000 mbar = 1 bar¹.

It is often convenient to introduce the "EXNER function"

$$\Pi = c_p \left(\frac{p}{p_0}\right)^{1-\frac{1}{\kappa}}. \quad (2.11)$$

One has

$$\alpha dp = \theta d\Pi, \quad (2.12)$$

$$T ds = \Pi d\theta \quad (2.13)$$

and

$$h = \Pi\theta + \text{constant}. \quad (2.14)$$

These formulas will be referred to later.

Dry air is a two-parameter system, the state of which is determined by the two independent functions of state, e.g. p and ρ . If in a mass of dry air the changes of state of the particles are *isentropic*, then the air is *piezotropic*, and the equation of piezotropy is obtained from (2.10), where θ is now a constant for each particle. Isentropic changes of state of dry air are referred to as

¹ The potential temperature is the temperature that the air would assume when brought adiabatically from its actual pressure to the pressure of 1000 mb.

dry-adiabatic. The corresponding coefficient of piezotropy is

$$\gamma_d = \left(\frac{D\varrho}{Dp} \right)_d = \frac{1}{\kappa R T} \quad (2.15)$$

and from (2.7) one obtains

$$\left(\frac{DT}{Dp} \right)_d = \frac{\alpha}{c_p}. \quad (2.16)$$

3. Thermodynamic properties of moist air. *a) Thermodynamics of the water component.* Water vapor does not follow the laws of an ideal gas nearly as accurately as does dry air. Nevertheless, it suffices for most purposes to consider also water vapor as an ideal gas. The errors introduced by this assumption will be small as far as the meteorological properties of moist air are concerned, since the amount of water in atmospheric air constitutes only a small fraction of the total mass.

Denoting the partial pressure and density of water vapor by p' and ϱ' , its equation of state may therefore be written

$$\varrho' = \sigma \frac{p'}{R T}. \quad (3.1)$$

Here R is the gas constant of dry air, and $\sigma = 0.622$ is the relative density of water vapor with respect to dry air. Similarly, the enthalpy of water vapor may be written

$$h' = c'_p T + \text{constant}, \quad (3.2)$$

where c'_p is the specific heat of water vapor at constant pressure.

Liquid water may be treated as incompressible. Denoting its specific heat by c'' , the enthalpy per unit mass may be written.

$$h'' = c'' T + \text{constant}. \quad (3.3)$$

By definition, the latent heat of evaporation is

$$L = h' - h'', \quad (3.4)$$

where h' and h'' refer to states of vapor and liquid water which are in equilibrium with each other. However, since both h' and h'' are assumed to be functions of T only, (3.4) may be considered to be true also for vapor and liquid water which are not in equilibrium, provided only that their temperature is the same. The enthalpy of water vapor may therefore be written

$$h' = c'' T + L + \text{constant}, \quad (3.5)$$

where the arbitrary constant is identic with the constant in (3.3). The heat of condensation varies with the temperature, but the variation is slow, and within the range of temperatures occurring in the atmosphere one may for most purposes consider L as a constant.

Let $P(T)$ denote the *saturation pressure* of water vapor with respect to liquid water with a plane interface. The details of the condensation processes in the atmosphere are beyond the scope of this article¹; it suffices here to assume that: 1. if the water component is present as vapor only, then $p' \leq P(T)$; and 2. if the air contains drops of liquid water, then $p' = P(T)$. Case 1. will be referred to as *unsaturated* (or *clear*) air, and case 2. as *saturated* (or *cloudy*) air.

¹ See the chapter by MASON and LUDLAM in this volume.

What is said above about the liquid phase is true for the solid phase also, and the same formulas apply, provided only c'' and L are interpreted as specific heat of ice and latent heat of sublimation, respectively.

β) *Unsaturated air*. Consider a mass of moist air (of pressure p and density ϱ) which is a mixture of dry air (of partial pressure p_a and partial density ϱ_a) and water vapor (of partial pressure p' and partial density ϱ'). Applying the laws for ideal gaseous mixtures, one has

$$p = p_a + p', \quad \varrho = \varrho_a + \varrho'. \quad (3.6)$$

The equation of state for the dry air component is

$$\varrho_a = \frac{p_a}{R T} \quad (3.7)$$

and for the water vapor component (3.1). The mixing ratio is defined as

$$\mu' = \frac{\varrho'}{\varrho_a} = \sigma \frac{p'}{p - p'}. \quad (3.8)$$

In unsaturated air, this quantity will change only as a result of diffusion and turbulent mixing. This is usually a slow process, and in many cases one may consider μ' as an individual constant so that

$$D\mu'/Dt = 0. \quad (3.9)$$

From (3.1), (3.6), (3.7) and (3.8) one derives

$$\frac{p}{\varrho} = \left(1 + \frac{1 - \sigma}{\sigma} \frac{\mu'}{1 + \mu'}\right) R T. \quad (3.10)$$

Hence, if the mixing ratio is constant, moist air satisfies the gas equation, and its gas constant is slightly larger than the gas constant of dry air. It is customary in meteorology to write this equation in the form

$$p/\varrho = R T_v, \quad (3.11)$$

where

$$T_v = \left(1 + \frac{1 - \sigma}{\sigma} \frac{\mu'}{1 + \mu'}\right) T \quad (3.12)$$

is called the *virtual temperature*. The difference $(T_v - T)$ may amount to a few degrees centigrade. If no great accuracy is needed, one may neglect the effect of humidity upon the equation of state and use the gas equation for dry air (2.1).

From (2.3) and (3.5), one obtains for the enthalpy per unit mass of moist air

$$h = \frac{c_p + \mu' c''}{1 + \mu'} T + \frac{\mu'}{1 + \mu'} L + \text{constant}. \quad (3.13)$$

Since μ' is a small quantity, of the order 1% in the lower troposphere and still smaller at greater heights, a suitable approximation for most purposes is

$$h \approx c_p T + \mu' L + \text{constant}. \quad (3.14)$$

In the same approximation, the internal energy is

$$e \approx c_v T + \mu' L + \text{constant}. \quad (3.15)$$

One may therefore write, for any process which satisfies (3.9),

$$T Ds = Dh - \alpha Dp \approx c_p DT - \alpha Dp. \quad (3.16)$$

Thus, nonsaturated air behaves in this approximation in the same way as dry air. In particular, the isentropic change of state will be very nearly unaffected by the humidity.

γ) Saturated air. Let a mass of moist air contain liquid water, so that one mass unit of dry air is connected with μ' mass units of water vapor and μ'' mass units of liquid water. The vapor is assumed to be in equilibrium with the liquid phase, so that

$$p' = P(T). \quad (3.17)$$

In this case, (3.8) yields

$$\mu' = \sigma \frac{P(T)}{p - P(T)} \approx \sigma \frac{P(T)}{p} \quad (3.18)$$

i.e. the mixing ratio in saturated air is a function of p and T . The quantity will change as a result of condensation and evaporation, so that (3.9) is no longer applicable. From (3.18) one obtains

$$D\mu' \approx \sigma \frac{dP}{p} \frac{dT}{dT} - \frac{\sigma P}{p^2} Dp. \quad (3.19)$$

The total mass of the water component per unit mass of dry air is

$$\mu = \mu' + \mu''. \quad (3.20)$$

In the absence of diffusion, turbulent mixing and precipitation, one has

$$D\mu/Dt = 0 \quad (3.21)$$

which now replaces (3.9).

The water drops will slightly increase the mean density of the air, and this effect can easily be expressed in terms of μ'' . Such a formula is of little use, because μ'' is not known. The effect is very small, however, since μ'' is always a small quantity, and one may therefore as in the case of nonsaturated air in most applications use the gas equation for dry air (2.1).

The enthalpy per unit mass of the two-phase system is

$$h = \frac{c_p + \mu c''}{1 + \mu} T + \frac{\mu'}{1 + \mu} L + \text{constant}. \quad (3.22)$$

As in the case of nonsaturated air, this formula may be replaced by the approximate formula

$$h \approx c_p T + \mu' L + \text{constant} \quad (3.23)$$

which may be used whether the air is saturated or not. However, in the saturated case, μ' is no longer an individual constant. From (3.23) and (3.19) one finds

$$T Ds = Dh - \alpha Dp \approx \left(c_p + \frac{\sigma L}{p} \frac{dP}{dT} \right) DT - \left(\alpha + \frac{\sigma L P}{p^2} \right) Dp. \quad (3.24)$$

The difference between the nonsaturated and the saturated case is apparent from a comparison between (3.16) and (3.24).

Isentropic changes of state of saturated air are referred to as *moist-adiabatic*. For such processes, we have according to (3.24)

$$\left(\frac{DT}{Dp} \right)_m \approx \frac{\alpha + \frac{\sigma L P}{p^2}}{c_p + \frac{\sigma L}{p} \frac{dP}{dT}}. \quad (3.25)$$

When its magnitude is compared with the dry-adiabatic change of state (2.16) one finds

$$\left(\frac{DT}{Dp}\right)_m < \left(\frac{DT}{Dp}\right)_d. \quad (3.26)$$

The difference between these quantities is appreciable for relatively high temperatures. However, since P and dP/dT tend rapidly towards zero as T decreases, it follows that $(DT/Dp)_m$ approaches $(DT/Dp)_d$ for low temperatures; and their difference is for most purposes negligible at temperatures below -10° Centigrade.

As a conclusion, we may say that atmospheric air may, as far as its thermodynamic properties are concerned, behave in two different ways: 1. Non-saturated (or clear) air behaves nearly like dry air, regardless of its humidity; and 2. saturated (or cloudy) air has properties which differ from those of non-saturated air, and which are nearly independent of the content of liquid water or ice. Therefore, the total content of the water constituent is of little importance as long as we know whether the air is saturated or not, and we may then consider the thermodynamic state of the air as given by two functions of state, e.g. pressure and density. However, knowledge of the total water content is essential if we want to determine when saturation will begin and when it will end.

In many theories in dynamic meteorology, the condensation processes are disregarded, and the air is considered as non-saturated, or dry.

For a more complete treatment of atmospheric thermodynamics, the reader is referred to the textbooks [1], [4], [9], [11], [12] and [16, I].

b) Basic equations of physical hydrodynamics.

4. The hydrodynamic equations. α) *The EULERIAN representation.* The EULERIAN representation, with three space coordinates and time as independent variables, is most commonly used in dynamic meteorology. The space coordinates may be CARTESIAN, spherical or any other system; however, the vector notation permits us to write equations without specifying the coordinate system.

The EULERIAN description of a hydrodynamic process involves the knowledge of the velocity vector \mathbf{v} and the thermodynamic state in all points and at all times. In most cases, the thermodynamic state will be considered as determined by two thermodynamic functions, e.g. pressure p and density ρ . If we want to take into account the effect of humidity, another variable, e.g. the mixing ratio μ must be added. Therefore the dependent variables, which are to be considered as functions of the space coordinates and time, are \mathbf{v} , p , ρ and possibly μ .

The operator D/Dt expressing individual differentiation with respect to time may be expanded in the following way

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad}, \quad (4.1)$$

where the first term on the right is referred to as the local rate of change, and the second term as the convective rate of change. This formula is applicable to vector fields as well as scalar fields; and it is valid in any frame of reference.

β) *The equation of continuity.* The vector $\rho\mathbf{v}$ represents the flux of mass; and the net flow of mass out of a resting volume element $d\tau$ per unit time is therefore $\text{div}(\rho\mathbf{v})d\tau$. According to the principle of conservation of mass, this expression must be equal to the rate of decrease of the mass contained within the

same volume element, which is $-\frac{\partial \varrho}{\partial t} d\tau$. Hence

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0. \quad (4.2) \quad \text{?}$$

This equation is called the *equation of continuity*. By means of (4.1) and the identity

$$\operatorname{div}(\varrho \mathbf{v}) = \varrho \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \varrho, \quad (4.3)$$

the equation of continuity may be written in the alternative form

$$\operatorname{div} \mathbf{v} = -\frac{1}{\varrho} \frac{D\varrho}{Dt} = \frac{1}{\alpha} \frac{D\alpha}{Dt}. \quad (4.4)$$

If the humidity is taken into account, we will need a corresponding continuity equation for the water component. However, no simple expression is obtained for the flux of water, since it will depend upon such processes as precipitation, evaporation and diffusion.

From the equation of continuity (4.2) and the differentiation rule (4.1), we may derive the following relation

$$\varrho \frac{D\psi}{Dt} = \frac{\partial(\varrho \psi)}{\partial t} + \operatorname{div}(\varrho \mathbf{v} \psi), \quad (4.5)$$

where ψ denotes any scalar or vector function of the space coordinates and time.

γ) *The equation of motion referred to an inertial frame.* The forces acting upon an air particle referred to unit mass, are the following: 1. The force of pure gravity $\mathbf{g}_a = -\operatorname{grad} \varphi_a$, where φ_a is the potential of the force of gravitational attraction; 2. the pressure force $-\alpha \operatorname{grad} p$; and 3. the force of molecular friction $\alpha \operatorname{div} \mathcal{F}$, where \mathcal{F} is the NAVIER-STOKES frictional stress tensor.

Denoting by \mathbf{v}_a the velocity relative to an inertial frame of reference, and by D_a the individual change relative to an inertial frame of reference, the acceleration is $D_a \mathbf{v}_a / Dt$. NEWTON's second law, written for a mass unit of air, therefore gives

$$D_a \mathbf{v}_a / Dt = -\operatorname{grad} \varphi_a - \alpha \operatorname{grad} p + \alpha \operatorname{div} \mathcal{F}. \quad (4.6)$$

This is the equation of motion referred to an inertial frame. According to (4.1), the acceleration may be written

$$\frac{D_a \mathbf{v}_a}{Dt} = \left(\frac{\partial \mathbf{v}_a}{\partial t} \right)_a + \mathbf{v}_a \cdot \operatorname{grad} \mathbf{v}_a, \quad (4.7)$$

where $(\partial \mathbf{v}_a / \partial t)_a$ is the local acceleration in the inertial frame, and $\mathbf{v}_a \cdot \operatorname{grad} \mathbf{v}_a$ the convective acceleration. An alternative expression for the convective acceleration which is often useful is

$$\mathbf{v}_a \cdot \operatorname{grad} \mathbf{v}_a = \operatorname{curl} \mathbf{v}_a \times \mathbf{v}_a + \operatorname{grad} \left(\frac{1}{2} v_a^2 \right). \quad (4.8)$$

If we let the frame of reference follow the earth in its orbital motion without participating in its rotation, Eq. (4.6) will hold provided the inertial force due to the acceleration in the orbital motion is added to the forces. This inertial force will nearly balance the gravitational attraction from the moon and the sun, however, their resultant being the small tidal forces. Therefore, if we neglect the tidal force, Eq. (4.6) will hold in a non-rotating frame centered at the center of the earth, provided φ_a is the potential of gravitational attraction from the earth only. We shall refer to motions relative to this coordinate system as "absolute".

δ) *The equation of motion referred to a frame rotating with the solid earth.* The earth rotates around its axis with an angular velocity $\Omega = 0.729 \times 10^{-4} \text{sec}^{-1}$. Applying the right-handed-screw convention of correspondence between positive direction of rotation and positive direction of translation along the axis, the angular velocity may be represented as a vector Ω pointing toward the celestial North Pole.

Let \mathbf{v} denote velocity of the air relative to a frame of reference rotating with the earth (which will be referred to as "*relative velocity*"); let \mathbf{r} denote position vector from the center of the earth, and \mathbf{r}_e the equator projection of \mathbf{r} . In the rotating frame, NEWTON'S second law still holds, provided two fictitious inertial forces are added, viz. the centrifugal force and the CORIOLIS force. The centrifugal force per unit mass is

$$\Omega^2 \mathbf{r}_e = \text{grad } \frac{1}{2} (\Omega \times \mathbf{r})^2$$

and the CORIOLIS force $-2\Omega \times \mathbf{v}$.

The centrifugal force and the force of the earth's gravitational attraction must be considered as a unity since only their sum is observable. This sum

$$\mathbf{g} = \mathbf{g}_a + \Omega^2 \mathbf{r}_e = -\text{grad } \varphi \quad (4.9)$$

may be called the force of apparent gravity, but is usually referred to simply as the force of gravity. Its potential

$$\varphi = \varphi_a - \frac{1}{2} (\Omega \times \mathbf{r})^2 \quad (4.10)$$

is called the *geopotential*.

When the centrifugal force and the CORIOLIS force are added on the right-hand side of (4.6), we obtain the equation of motion in the rotating frame,

$$\frac{D\mathbf{v}}{Dt} = -\text{grad } \varphi - 2\Omega \times \mathbf{v} - \alpha \text{grad } p + \alpha \text{div } \mathcal{F}, \quad (4.11)$$

where D denotes individual change relative to the rotating frame.

By means of (4.1), the acceleration may be written as

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \text{grad } \mathbf{v} \quad (4.12)$$

where the terms on the right are the local and convective accelerations in the frame rotating with the earth.

Eq. (4.11) is referred to a mass unit. The equation of motion referred to a unit volume is obtained by multiplication with ρ . Making use of (4.5), we obtain the equation of motion in the equivalent form

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \text{div}(\rho \mathbf{v} \mathbf{v} + p\mathcal{E} - \mathcal{F}) = -\rho \text{grad } \varphi - 2\Omega \times \rho \mathbf{v} \quad (4.13)$$

where \mathcal{E} denotes the unit tensor. The first term on the left is the rate of increase of impulse per unit volume; the second term is the divergence of a tensor which may be interpreted as an impulse flux, and the right-hand side represents the rate of production of impulse.

ε) *Boundary conditions.* On a rigid boundary surface, $F(x, y, z) = 0$, the normal velocity component must vanish; i.e.

$$\mathbf{v} \cdot \text{grad } F = 0 \quad \text{when} \quad F(x, y, z) = 0. \quad (4.14)$$

This is the only condition to be fulfilled at a rigid boundary in a nonviscid fluid.

In a viscous fluid, also the tangential velocity component must vanish, so that

$$\mathbf{v} = 0 \quad \text{when} \quad F = 0. \quad (4.15)$$

These conditions are easily generalized to the case of a boundary surface of specified motion.

There occur in the atmosphere abrupt transition zones (fronts and inversions) which are most conveniently treated as internal *surfaces of discontinuity*. On such surfaces, the equations of motion and continuity lose their meaning, because the derivatives involved do not exist; therefore they must be replaced by interface conditions.

Suppose that $F(x, y, z, t) = 0$ is the equation of the interface (which is usually unknown); and suppose further that the interface is a material surface. Letting subscripts 1 and 2 indicate the dependent variables on either side of the interface, we have

$$\left. \begin{aligned} \frac{\partial F}{\partial t} + \mathbf{v}_1 \cdot \text{grad } F &= 0 \\ \frac{\partial F}{\partial t} + \mathbf{v}_2 \cdot \text{grad } F &= 0 \end{aligned} \right\} \quad \text{when } F = 0. \quad (4.16)$$

These equations are called the *kinematic interface conditions*; they replace the continuity equation at the interface. When these equations are subtracted, one obtains

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \text{grad } F = 0 \quad \text{when } F = 0 \quad (4.17)$$

showing that the normal velocity components are the same on both sides. This equation is not sufficient, however, since F is not known, and (4.17) must be supplemented by one of the Eqs. (4.16); this would suffice in the non-viscid case. In the case of viscous fluids, one has

$$\left. \begin{aligned} \mathbf{v}_1 &= \mathbf{v}_2 \\ \frac{\partial F}{\partial t} + \mathbf{v} \cdot \text{grad } F &= 0 \end{aligned} \right\} \quad \text{when } F = 0. \quad (4.18)$$

The equations of motion must be replaced by the dynamic interface condition. This is, in the case of a nonviscous fluid,

$$p_1 - p_2 = 0 \quad \text{when } F = 0. \quad (4.19)$$

For a viscous fluid, one must write instead

$$\text{grad } F \cdot [-(p_1 - p_2)\mathcal{E} + \mathcal{F}_1 - \mathcal{F}_2] = 0 \quad \text{when } F = 0. \quad (4.20)$$

5. Circulation and vorticity. α) *Circulation theorem of V. BJERKNES in absolute motion.* Consider an arbitrary closed curve L within the fluid, with a chosen positive direction along it. Neglecting frictional forces and taking the line integral of (4.6) in positive direction along L , the force of gravity drops out and one obtains

$$\oint_L \frac{D_a \mathbf{v}_a}{Dt} \cdot d\mathbf{r} = - \oint_L \alpha \text{grad } p \cdot d\mathbf{r}. \quad (5.1)$$

If the closed curve is defined as a material curve, which follows the air in its motion, then (5.1) may be written

$$\frac{dC_a}{dt} = \oint_L (-\alpha \text{grad } p) \cdot d\mathbf{r} = - \oint_L \alpha dp, \quad (5.2)$$

where

$$C_a = \oint_L \mathbf{v}_a \cdot d\mathbf{r} \quad (5.3)$$

is called the *absolute circulation*. Eq. (5.2) is the *first circulation theorem* of V. BJERKNES¹. By means of the theorem of STOKES, (5.2) may also be written

$$\frac{dC_a}{dt} = \int_\sigma \mathbf{S} \cdot d\boldsymbol{\sigma}, \quad (5.4)$$

where $d\boldsymbol{\sigma}$ is the vectorial surface element of any surface σ which has L as boundary curve, and where

$$\mathbf{S} = \text{curl}(-\alpha \text{ grad } p) = -\text{grad } \alpha \times \text{grad } p \quad (5.5)$$

will be referred to as the "*baroclinicity vector*".

The field of \mathbf{S} is non-divergent or "solenoidal". A convenient representation of any vector field with this property is obtained by dividing the space into tubes or "solenoids" of unit vector flux. A set of such solenoids for the field \mathbf{S} is formed by the family of isosteric surfaces $\alpha = \text{constant}$ (spaced one volume unit apart) and the family of isobaric surfaces $p = \text{constant}$ (spaced one pressure unit apart). These solenoids will have parallelogrammatic cross-section; they may be given positive direction in the direction of \mathbf{S} .

The right-hand side of (5.5) is seen to represent the number of such isosteric-isobaric solenoids embraced by the curve L ; the number is to be counted algebraically, so that a solenoid counts $+1$ or -1 according to whether it runs through the curve L (through the surface σ) in positive or negative direction.

By means of (2.6) and (2.12), the baroclinicity vector \mathbf{S} may also be written

$$\mathbf{S} = -\text{grad } s \times \text{grad } T = -\text{grad } \theta \times \text{grad } \Pi \quad (5.6)$$

and alternative solenoids of \mathbf{S} are therefore obtained from the isentropic surfaces $s = \text{constant}$ and the isothermal surfaces $T = \text{constant}$, or from the surfaces $\theta = \text{constant}$ and $\Pi = \text{constant}$.

According to V. BJERKNES², the density field is said to be *barotropic* if $\mathbf{S} = 0$ in every point within a region. This implies that the isosteric surfaces $\alpha = \text{constant}$ coincide with the isobaric surfaces $p = \text{constant}$, or that $\text{grad } \alpha = 0$ (the case $\text{grad } p = 0$ is uninteresting). In other words, the barotropic state is characterized by a constant density in each isobaric surface. The *coefficient of barotropy* Γ is defined by

$$\text{grad } \rho = \Gamma \text{ grad } p. \quad (5.7)$$

Within a barotropic region, the circulation theorem takes the form

$$dC_a/dt = 0. \quad (5.8)$$

On the other hand, the density field is said to be *baroclinic* if $\mathbf{S} \neq 0$. This implies that $\text{grad } \alpha$ and $\text{grad } p$ are both different from zero and that the isosteric surfaces do not coincide with the isobaric surfaces. Hence, baroclinicity means that the density distribution in each isobaric surface is non-uniform.

If the fluid remains in the state of rest, then the circulation of every material closed curve remains zero, and it follows that the state must be barotropic. Such a state of rest can be disturbed by means of heat and cold sources, provided the fluid is not piezotropic. The density in an isobaric surface will decrease when the fluid is heated and increase where the fluid is cooled, so that the initially uniform

¹ V. BJERKNES: Vidensk. selsk. Skr., Christiania (Oslo), Math.-naturv. Kl. 1898, No. 5. — Sv. Vetensk. Akad. Handl., Stockh. 31, Nr. 4 (1898).

² V. BJERKNES: Geof. Publ. 2, No. 4 (1921).