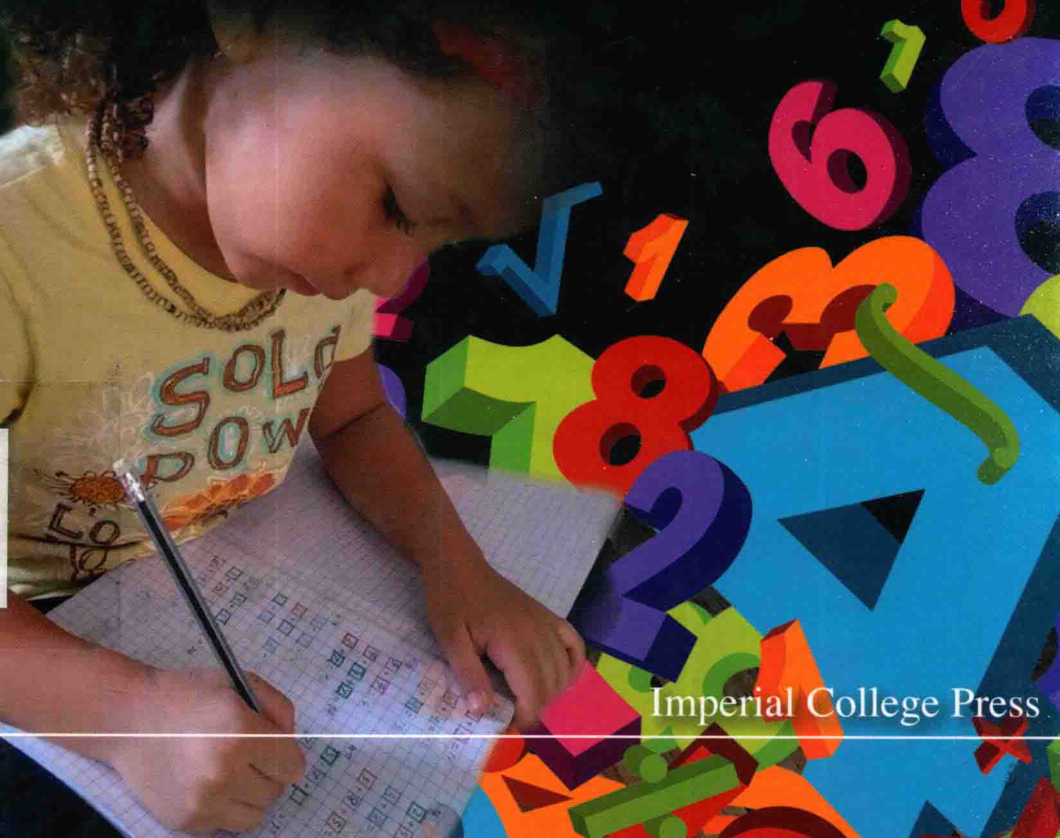
Ana Helvia Quintero • Héctor Rosario

Math Makes Sense!

A Constructivist Approach to the Teaching and Learning of Mathematics



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PREFACE[1]

Experience has taught us that a great number of students do not learn the concepts of mathematics that they are taught in elementary school. Nevertheless, these same children use mathematics in their everyday environment. For example, without knowing anything about the formal theory of fractions or rational numbers, children know how to "fairly" divide a pizza or a chocolate bar. This book shows us how to capitalize on this informal knowledge of mathematics that children bring with them, and build upon it the corresponding mathematical concepts in a formal way.

Through examples and activities, the book shows paths for learning with understanding the mathematical concepts taught in elementary school. While discussing these paths, we emphasize that many of the concepts that are taught are much more difficult than some educators might think. Hence, we focus on less content and more depth, based on our teaching philosophy that children should be taught basic concepts with a thorough understanding of what they mean, instead of piling up material with no comprehension.

This book also presents activities that foster the learning of mathematical concepts in an environment of exchange that promotes an interest in solving problems and making conjectures, as well as in sharing, discussing, and

[1] This is a revised and expanded translation of the book *Matemáticas con sentido: aprendizaje y enseñanza* by Ana H. Quintero, published by La Editorial Universidad de Puerto Rico. It includes many new illustrations and an additional chapter on mathematical puzzles.

defending ideas with classmates. An environment that encourages student participation — in which students develop basic skills and learn to understand and appreciate the language of mathematics — has as its final goal the promise of enabling students to enter the wonderful world of mathematical creation and imagination.

Yet, regardless of how perfect our presentation of the devolvement of these concepts may be, we will never capture the richness and diversity that are found in the classroom. Therefore, along with the paths we offer, it is necessary to create a school (or home) environment that promotes constant reflection in students and teachers. In this connection, we present ideas in the form of research questions that should prove stimulating for teachers, who are responsible for kindling students' interest.

ACKNOWLEDGMENTS

This book is the product of years of research. Many people have contributed throughout this process and I am grateful to all of them. I especially appreciate the assistance of Dr. Jorge M. López, who introduced me to the field of realistic mathematics education (RME). I have shared with him throughout the years my research as well as my inner thoughts on how mathematics is learned. The CRAIM team (Centros Regionales de Adiestramiento en Instrucción Matemática), led by Jorge, has also been actively involved in this process.

Many teachers from numerous schools have shared their experiences in the classroom with me through active collaboration. Their work has greatly enhanced the experiences presented in this book. In particular, I deeply appreciate the support provided by the teachers and principals at the following elementary schools: Antonio S. Pedreira in Puerto Nuevo, Abraham Lincoln in Old San Juan, Sofía Rexach in Cantera, and Rafael Cordero in Cataño.

I have shared different versions of the various chapters included in this book with Professor Luis López, from the Faculty of Education at the University of Puerto Rico, who used them for his "Methodology of Teaching Mathematics" course. I thank him and his students for their support in the process of clarifying the arguments and explanations presented in those chapters.

The Mathematics Department of the Faculty of Natural Sciences at the Río Piedras Campus of the University of Puerto Rico was always willing to give me the support and time to develop this project. I also thank La Editorial Universidad de Puerto Rico for granting permission to use the illustrations from the Spanish edition. Finally, I appreciate the support for my endeavors given by my family. They shared with me not only the teachings and reflections of this project, but also sustained me emotionally.

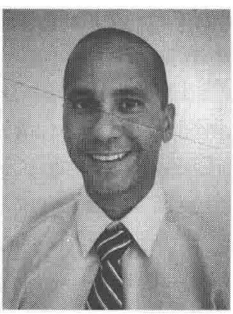
AHQ

I would like to express my appreciation for the translating assistance provided by my friend Pedro Zayas (Chapters 2–7). My wife Verónica del Mar and oldest daughter Ishtar also helped me with translating work (Acknowledgements, Preface, and Chapter 8). My family has borne the burden of this rapid translation, revision, and expansion of a book highly regarded by elementary mathematics teachers and teacher trainers in Puerto Rico. I also thank my homeschooled children — Arjuna, Rohini, Vrinda, Haridas, and Krishna — for being my greatest teachers while playing the role of students during my experiments with the teaching of mathematics. My approach with them is sprinkled throughout the book. I am deeply thankful for my family's love and support.

HR

ABOUT THE AUTHORS

Ana Helvia Quintero is Professor of Mathematics at the University of Puerto Rico in Río Piedras and holds a Ph.D. in the learning of mathematics from MIT. Her main research interest is mathematics education and has been involved in several projects aimed at improving the school system in Puerto Rico. From January 2001 to June 2002, she was Under Secretary of Education of the Commonwealth of Puerto Rico. Based on that experience, she published a book in 2006, *Muchas reformas, pocos cambios* (Many Reforms, Hardly Any Changes).



Héctor Rosario holds a Ph.D. in Mathematics education from Columbia University. During the production of this book, he was Professor of Mathematics at the University of Puerto Rico at Mayagüez; he is now a faculty member at the North Carolina School of Science and Mathematics. Dr. Rosario is the principal editor of the 17-nation anthology *Mathematics and Its Teaching in the Southern Americas*, published by World Scientific Publishing. He is also a member of the advisory board of the National Association of Math Circles.

CONTENTS

CHAPTER 1

FOSTERING THE LEARNING OF MATHEMATICS

1.1 Introduction

Mathematics is one of the most difficult subjects for many students. A great deal of this difficulty is a result of how it is taught. The difficulties of learning mathematics can be grouped into two major categories: issues that are similar to those of teaching other subjects — it is taught without understanding and without taking into consideration how it is learned (Bruner, 1990; Cohen *et al.*, 1993; Quintero *et al.*, 2006; Wiske, 1998) — and those that spring from the way in which the teaching of the mathematical concepts is organized. In this book we will discuss both types of issues. However, since the literature is vast concerning the difficulties shared with other subjects, we will only summarize them and recommend some further reading, while emphasizing the problems that spring from how mathematical concepts are organized for teaching. We understand that there is much research to be done on this aspect. In this regard, this book presents the status of current research while intending to promote future studies.

1.2 Problems in the Teaching of Mathematics Shared with Other Disciplines

1.2.1 *Teaching without understanding*

We all remember from our school days some instances when we were taught without understanding. We recall the act of having learned, but not what was taught. Whatever is learned in this way will be forgotten easily and will not contribute to the understanding of our world, either natural or cultural. Every day, research on the teaching for understanding increases (see, for instance, Bruner, 1990; Cohen *et al.*, 1993; Quintero *et al.*, 2006; Wiske, 1998). When learning is relevant, we can apply our knowledge to new situations and make connections among diverse fields of knowledge; we strengthen our competence to use our knowledge (Bransford *et al.*, 2000) and we are able to connect our knowledge to our day-to-day experiences.

Children show interest in mathematical ideas form an early age (McCrink and Wynn, 2004; Whalen *et al.*, 1999). Through their day-to-day experience, children develop informal ideas about numbers, quantities, patterns, shapes, and size, among other concepts. The learning of mathematical ideas begins much earlier than children's formal school experience (Gelman and Gallistel, 1978; Resnick, 1987). Even after entering school, students of all ages develop mathematical ideas on a daily basis (Bransford *et al.*, 2000). A problem with contemporary teaching is that it does not integrate these experiences into the formal classroom learning environment. When one of the authors was working on verbal problems with fifth graders (10–11 years old), she had the following exchange with them after posing this problem:

> *A man bought 20 oranges. If the oranges are 5 for a dollar, how much did he spend?*
>
> *Student: I don't know how to do the problem. I know the answer is $4.00, but I don't know how to do it.*
>
> *Researcher: But, how do you know that the answer is $4.00?*
>
> *Student: Well, if they give 5 oranges for a dollar, for $2.00 they'll give 10, so that 20 oranges will be $4.00.*

We see that students separate two types of knowledge they possess to solve the problem. On the one hand, they have the informal learning of day-to-day life in and out of school. On the other, they carry out the arithmetic operations, which they do not relate to their informal knowledge. They think that solving a problem equals translating it into a single arithmetic operation (addition, subtraction, multiplication, or division), as is demanded from them at school. Being unable to do this translation, they feel that they cannot solve the problem, though they have done so with their informal knowledge.

It is imperative to bridge the informal math that students do daily with the formal math taught in schools. In fact, the math that students learn through their daily interactions is learned with meaning and can be used in diverse contexts, as shown in the example above. The math they learn in school is for the most part a set of meaningless rules that they barely use outside of school. An area of research that college professors and school teachers must share is that of identifying activities that, springing from the students' experiences, support the construction of mathematical concepts.

Mathematics is one of the most misunderstood disciplines. The understanding among a great portion of the population is that mathematics is a series of rules used for numerical calculations. Responding to this interpretation, mathematics is taught as a set of formulas to do various calculations. Mathematics, however, as most fields of knowledge, developed as a result of the human need to understand and interpret the world (Davis and Hersh, 1972; Kline, 1973). The teaching of mathematics must therefore spring from contexts that are meaningful for the students and build on their informal knowledge. For example, if we want to introduce the concept of measurement in second grade (ages seven to eight years), we can start with the question, "How much do we grow in a year?" We start by discussing with students their notions of measurement, and from these build up to the more sophisticated notions that mathematics has developed.

Throughout the book, we present meaningful contexts for teaching various mathematical concepts. Teachers, in turn, will investigate and, based on their research, develop a database of examples of contexts that promote the learning of the mathematical concepts taught. In fact, while teaching most curricular subjects, situations arise that lend themselves to mathematical analysis. For example, when studying the human body in kindergarten, we may ask, "How many eyes?", "How many hands?", or "How many fingers?" Ideally, the teaching of mathematics should be integrated into the teaching of other subjects. If during social studies class we study the community, we can use that context — the corner shop or children in different areas of the community — to present counting problems, as well as addition and subtraction problems (see Fundación Quintero Alfaro, 1993, 1994, for examples of schools in Puerto Rico working with curriculum integration).

1.2.2 *Teaching as information transfer*

Teaching based on meaningful contexts allows students to connect what they are learning with their ideas and experience. But it's not enough just to start from meaningful contexts, it is also important to promote active and constructive learning. The most widespread misconception of learning is that children come into this world as a *tabula rasa*, i.e. their minds contain no ideas or thoughts and that they gain knowledge from their experience and what they receive from adults. In the 1950s, Swiss biologist Jean Piaget (1921, 1966, 1976) presented a different interpretation of how humans learn. He stated that human beings, rather than being processors of information, are creators of models, which they use to understand the world and to theorize about it.

Annette Karmiloff-Smith and Bärbel Inhelder (1975) suggest that children develop theories to explain the world from a very early age. Originally these theories, which they call *theories in action*, are composed of some implicit ideas or representation models of a situation. In their interactions with the physical and cultural world, humans develop their theories. Piaget noted that children did not repeat the explanations presented by adults of an event, but rather built their own from the connections made between their experience and their theories. For instance, until about nine years of age, the model that children use to explain the behavior of living beings and inanimate objects is the model of human behavior in society. Thus, a child before this age will explain the motion of the moon from this model and say, for example, that the moon went for a walk or that it was angry and hid, even after hearing the explanation of the motion of the moon based on physical laws. Children do not merely repeat the explanations offered, but construct their interpretations from their own theories.

Piaget's theory about the process of constructing knowledge has been put forward, independently, by other scholars, such as Vygotsky (1978) and Luria (1976). These theorists emphasize the role of culture in the construction made by the child. From these great thinkers, more recent researchers have developed the constructivist learning theory (Bruner, 1996; Lakoff, 1987).

Constructivism suggests that humans are seeking to understand the world around them. In this quest, they are active entities exploring their environment. During their search, they develop schemes of understanding that expand according to their interactions with their physical and cultural environments. Thus, when confronted with a new experience, they can accommodate it in their current scheme, or revise and extend that scheme to integrate the new experience. For example, the original food scheme of a baby is to suck. At the beginning, babies only drink milk. The first time they take juice, they might seem surprised, but easily accommodate the new experience into their sucking scheme. Now when they are given solid food, they try sucking and make a mess. Gradually, they adapt their scheme and extend it to sucking when the food is liquid and swallowing when it is solid. These processes are what Piaget called accommodation and adaptation. When their mental schemes are not ready to integrate a situation, people may misinterpret the situation by integrating it into one of their existing schemes, or may simply overlook the situation. The example of a child's interpretation of the motion of the moon is an example of the first case.

In the teaching of any discipline it is important to provide the opportunity for students to explore, make conjectures, discuss them, and, gradually, actively build their knowledge. We introduce the context, and in it the question or problem that leads to the subject we want to address. Then we allow students to develop their own solutions. Students can work individually or in groups. Once students have worked on solving the problem, both correct and incorrect strategies are discussed. Through questions, we promote students' learning from each other. Allowing students to build their solutions enables them to build their own mathematical knowledge. This classroom dynamic provides an opportunity for the teacher to identify the students' concepts and ideas, and thus support their construction upon their conceptions. For instance, let us assume that we are in the first few weeks of school. In fifth grade (10–11 years), we are interested in developing the division algorithm. In the first few days, we meet with parents and take advantage of this to present the following problem:

> *81 people will be present at the Parent Teacher Association meeting. If we can accommodate 6 people per table, how many tables do we need?*

In a study conducted in a classroom we found the following strategies for solving this problem:

— Add $6 + 6 + 6 + \cdots$
— Count 6, 12, 18, ...
— Start from 81 and subtract 6 many times
— Say the times table $1 \times 6, 2 \times 6, 3 \times 6, \ldots$ until we reach the product closest to 81.

An analysis of these responses shows that the child who adds $6 + 6 + 6$, etc., does not interpret the above problem in a multiplicative structure. Prior to introducing division we have to develop the idea of multiplication. The work of the fourth child, meanwhile, shows that he sees the multiplicative structure of the problem. In this case, we can introduce the concept of division based on his knowledge of multiplication.

We then introduce other problems with the structure of division, for example:

> *At the meeting we want to give parents soda. We decide on 2-L bottles. If each bottle serves 8 people, how many do we need to buy?*

After several problems with the same structure, we invite students to identify the common elements in them. From those methods they develop the algorithm. In Chapter 5, we will discuss the cognitive development of division and how to go about building the algorithm.

The process we have discussed in the example of the division problem must guide us in the conceptualization of mathematical terms. Thus, by introducing a problem or situation, we allow students to create their own strategies. This helps students to build their knowledge from their informal knowledge. Moreover, it allows teachers to create activities based on students' conceptions, which will serve to broaden, deepen, or correct them. Therefore, it is important for teachers to be attentive to the students' output, hear and observe their explanations, and foster reflection on them. In this process, the teacher directs students in analyzing their conceptions (Lampert 1989; Mack, 1990). Teachers should give room for students to discuss among themselves their strategies, thus learning from others as everyone explains their reasoning.

The active and constructive teaching process takes longer than the transfer of knowledge method. It is much easier to explain the division algorithm than it is to assist students in the construction of the elements that comprise it. Ultimately, however, the time saved is not worth it because students do not really learn, and even though they can pass an exam by mechanically applying information they have learned, that information will be easily forgotten because there was no integration into their conceptions and the material must be retaught. That is why we still find college students who do not understand fractions.

1.2.3 *Teaching without reflection*

In mathematics, concepts arise from reflecting upon mathematical activity. When we ponder different situations, we discover similarities and common elements that are abstract paradigms and developed into mental action patterns. Such patterns, similarities, and common elements are organized and systematized in mathematical structures. Once students have considered several situations that integrate a mathematical concept, they can abstract the mathematical structure through reflection. For example, once students have worked several problems with the same structure, these are discussed so as to identify common elements. At this point, symbolism, algorithms, and mathematical structures begin to be developed. For example, after working several problems with the structure of division — as those presented in the previous section — students realize that to find how many groups of the same quantity fit into a number (division model), it is more efficient to multiply than to repeatedly add the number of objects in the group. They can then find the major product of the divisor that fits into the dividend. From this pattern, a scheme emerges that includes the students' strategies, which in turn approaches the division algorithm. We see that the abstraction process should be gradual. We start with a concrete situation, which varies according to the level. For example, for students analyzing division problems for the first time, the algorithm of this operation is abstract. Once the student has experience with division, they can use it in the construction of other operations.

From the analysis of "concrete" situations, students create their own nomenclature to represent them. This is the pre-formal stage in which students introduce symbols, drawings, and diagrams that help them in the qualitative analysis of the situation. From the symbols, algorithms, strategies, and models that students make, we develop the mathematical language and symbols. This process takes longer than if we simply present and explain mathematical symbols. In the long run, however, we recover this time as students actually learn and we don't have to be in endless remedial sessions.

Hence, in each level we follow this pattern:

concrete → pre-formal → formal

For instance, while developing subtraction we have:

Concrete

Mary has 5 lollipops and gives 2 to Lilly. How many is she left with?