PLANE TRIGONOMETRY AND TABLES

GRANVILLE

PLANE TRIGONOMETRY
AND FOUR-PLACE TABLES OF LOGARITHMS

BY

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PREFACE

It has been the author's aim to treat the subject according to the latest and most approved methods. The book is designed for the use of colleges, technical schools, normal schools, secondary schools, and for those who take up the subject without the aid of a teacher. Special attention has been paid to the requirements of the College Entrance Board. The book contains more material than is required for some first courses in Trigonometry, but the matter has been so arranged that the teacher can make such omissions as will suit his particular needs.

The trigonometric functions are defined as ratios; first for acute angles in right triangles, and then these definitions are extended to angles in general by means of coördinates. The student is first taught to use the natural functions of acute angles in the solution of simple problems involving right triangles. Attention is called to the methods shown in §§ 23–29 for the reduction of functions of angles outside of the first quadrant. In general, the first examples given under each topic are worked out, making use of the natural functions. A large number of carefully graded exercises are given, and the processes involved are summarized into working rules wherever practicable. Illustrative examples are worked out in detail under each topic.

Logarithms are introduced as a separate topic, and attention is called to the fact that they serve to minimize the labor of computation. Granville's *Four-Place Tables of Logarithms* is used. While no radical changes in the usual arrangement of logarithmic tables have been made, several improvements have been effected which greatly facilitate logarithmic computations. Particularly important is the fact that the degree of accuracy which may be expected in a result found by the aid of these tables is clearly indicated. Under each case in the solution of triangles are given two complete sets of examples, — one in which the angles are expressed in degrees and minutes, and another in which the angles are expressed in degrees and the decimal part of a degree. This arrangement, which is characteristic of this book, should be of great advantage to those secondary schools in which college preparation involving both systems is necessary.

To facilitate the drawing of figures and the graphical checking of results a combined ruler and protractor of celluloid is furnished with each copy of the book, and will be found on the inside of the front cover.

The author's acknowledgments are due to Mr. L. E. Armstrong for verifying the answers to the problems, and to Mr. S. J. Berard for drawing the figures.

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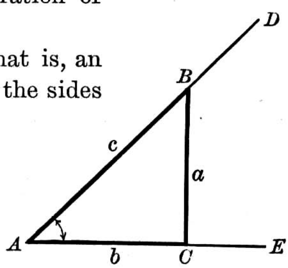
PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES
SOLUTION OF RIGHT TRIANGLES

1. Trigonometric functions of an acute angle defined. We shall assume that the student is familiar with the notion of the *angle between two lines* as presented in elementary Plane Geometry. For the present we will confine ourselves to the consideration of acute angles.



Let EAD be an angle less than 90°, that is, an acute angle. From B, any point in one of the sides of the angle, draw a perpendicular to the other side, thus forming a right triangle, as ABC. Let the capital letters A, B, C denote the angles and the small letters a, b, c the lengths of the corresponding opposite sides in the right triangle.* We know in a general way from Geometry that the sides and angles of this triangle are mutually dependent. Trigonometry begins by showing the exact nature of this dependence, and for this purpose employs the *ratios of the sides*. These ratios are called *trigonometric functions*. The six trigonometric functions of any acute angle, as A, are denoted as follows:

sin A,	read	"sine of A";
cos A,	read	"cosine of A";
tan A,	read	"tangent of A";
csc A,	read	"cosecant of A";
sec A,	read	"secant of A";
cot A,	read	"cotangent of A."

* Unless otherwise stated the hypotenuse of a right triangle will always be denoted by c and the right angle by C.

These trigonometric functions (ratios) are defined as follows (see figure):

(1) $\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}}\left(=\dfrac{a}{c}\right)$; (4) $\csc A = \dfrac{\text{hypotenuse}}{\text{opposite side}}\left(=\dfrac{c}{a}\right)$;

(2) $\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}}\left(=\dfrac{b}{c}\right)$; (5) $\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent side}}\left(=\dfrac{c}{b}\right)$;

(3) $\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}}\left(=\dfrac{a}{b}\right)$; (6) $\cot A = \dfrac{\text{adjacent side}}{\text{opposite side}}\left(=\dfrac{b}{a}\right)$.

The essential fact that the numerical value of any one of these functions depends upon the *magnitude* only of the angle A, that is, is independent of the point B from which the perpendicular upon the other side is let fall, is easily established.*

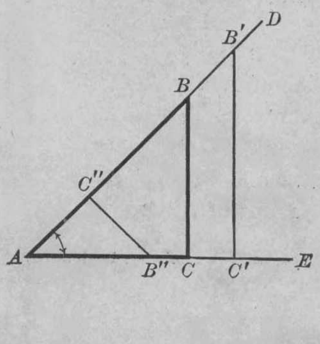
These functions (ratios) are of fundamental importance in the study of Trigonometry. In fact, no progress in the subject is possible without a thorough knowledge of the above six definitions. They are easy to memorize if the student will notice that the three in the first column are reciprocals respectively of those directly opposite in the second column. For,

$$\sin A = \frac{a}{c} = \frac{1}{\frac{c}{a}} = \frac{1}{\csc A}; \qquad \csc A = \frac{c}{a} = \frac{1}{\frac{a}{c}} = \frac{1}{\sin A};$$

$$\cos A = \frac{b}{c} = \frac{1}{\frac{c}{b}} = \frac{1}{\sec A}; \qquad \sec A = \frac{c}{b} = \frac{1}{\frac{b}{c}} = \frac{1}{\cos A};$$

$$\tan A = \frac{a}{b} = \frac{1}{\frac{b}{a}} = \frac{1}{\cot A}; \qquad \cot A = \frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{\tan A}.$$

* For, let B' be any other point in AD, and B'' any point in AE. Draw the perpendiculars $B'C'$ and $B''C''$ to AE and AD respectively. The three triangles ABC, $AB'C'$, $AB''C''$, are mutually equiangular since they are right-angled and have a common angle at A. Therefore they are similar, and we have



$$\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''}.$$

But each of these ratios defines the *sine of A*. In the same manner we may prove this property for each of the other functions. This shows that the size of the right triangle we choose is immaterial; it is only the relative and not the actual lengths of the sides of the triangle that are of importance.

The student should also note that every one of these six ratios will change in value when the angle A changes in size.

If we apply the definitions (1) to (6) inclusive to the acute angle B, there results

$$\sin B = \frac{b}{c}; \qquad \csc B = \frac{c}{b};$$

$$\cos B = \frac{a}{c}; \qquad \sec B = \frac{c}{a};$$

$$\tan B = \frac{b}{a}; \qquad \cot B = \frac{a}{b}.$$

Comparing these with the functions of the angle A, we see that

$$\sin A = \cos B; \qquad \csc A = \sec B;$$

$$\cos A = \sin B; \qquad \sec A = \csc B;$$

$$\tan A = \cot B; \qquad \cot A = \tan B.$$

Since $A + B = 90°$ (i.e. A and B are complementary) the above results may be stated in compact form as follows:

Theorem. *A function of an acute angle is equal to the co-function* of its complementary acute angle.*

Ex. 1. Calculate the functions of the angle A in the right triangle where $a = 3$, $b = 4$.

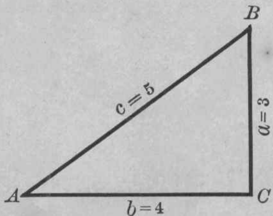
Solution. $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$

Applying (1) to (6) inclusive (p. 2),

$$\sin A = \tfrac{3}{5}; \qquad \csc A = \tfrac{5}{3};$$

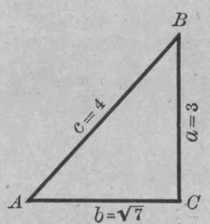
$$\cos A = \tfrac{4}{5}; \qquad \sec A = \tfrac{5}{4};$$

$$\tan A = \tfrac{3}{4}; \qquad \cot A = \tfrac{4}{3}.$$



Also find all functions of the angle B, and compare results.

Ex. 2. Calculate the functions of the angle B in the right triangle where $a = 3$, $c = 4$.



Solution. $b = \sqrt{c^2 - a^2} = \sqrt{16 - 9} = \sqrt{7}.$

$$\sin B = \frac{\sqrt{7}}{4}; \qquad \csc B = \frac{4}{\sqrt{7}};$$

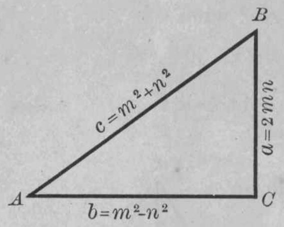
$$\cos B = \frac{3}{4}; \qquad \sec B = \frac{4}{3};$$

$$\tan B = \frac{\sqrt{7}}{3}; \qquad \cot B = \frac{3}{\sqrt{7}}.$$

Also find all functions of the angle A, and compare results.

* Sine and cosine are called co-functions of each other. Similarly tangent and cotangent, also secant and cosecant, are co-functions.

Ex. 3. Calculate the functions of the angle A in the right triangle where $a = 2\,mn$, $b = m^2 - n^2$.



Solution.

$$c = \sqrt{a^2 + b^2} = \sqrt{4\,m^2n^2 + m^4 - 2\,m^2n^2 + n^4}$$
$$= \sqrt{m^4 + 2\,m^2n^2 + n^4} = m^2 + n^2.$$

$$\sin A = \frac{2\,mn}{m^2 + n^2}; \qquad \csc A = \frac{m^2 + n^2}{2\,mn};$$
$$\cos A = \frac{m^2 - n^2}{m^2 + n^2}; \qquad \sec A = \frac{m^2 + n^2}{m^2 - n^2};$$
$$\tan A = \frac{2\,mn}{m^2 - n^2}; \qquad \cot A = \frac{m^2 - n^2}{2\,mn}.$$

Ex. 4. In a right triangle we have given $\sin A = \frac{4}{5}$ and $a = 80$; find c.

Solution. From (1), p. 2, we have the formula

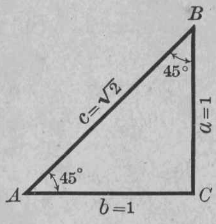
$$\sin A = \frac{a}{c}.$$

Substituting the values of $\sin A$ and a that are given, there results

$$\frac{4}{5} = \frac{80}{c};$$

and solving, $c = 100.$ *Ans.*

2. Functions of 45°, 30°, 60°. These angles occur very frequently in problems that are usually solved by trigonometric methods. It is therefore important to find the values of the trigonometric functions of these angles and to memorize the results.



(*a*) *To find the functions of 45°.*

Draw an isosceles right triangle, as ABC. This makes

$$\text{angle } A = \text{angle } B = 45°.$$

Since the relative and not the actual lengths of the sides are of importance, we may assign any lengths we please to the sides satisfying the condition that the right triangle shall be isosceles.

Let us choose the lengths of the short sides as unity, i.e. let $a = 1$ and $b = 1$.

Then $c = \sqrt{a^2 + b^2} = \sqrt{2}$, and we get

$$\sin 45° = \frac{1}{\sqrt{2}}; \qquad \csc 45° = \sqrt{2};$$
$$\cos 45° = \frac{1}{\sqrt{2}}; \qquad \sec 45° = \sqrt{2};$$
$$\tan 45° = 1; \qquad \cot 45° = 1.$$

(*b*) *To find the functions of 30° and 60°.*

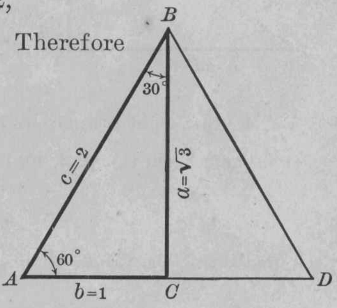
Draw an equilateral triangle, as ABD. Drop the perpendicular BC from B to AD, and consider the triangle ABC, where

$$\text{angle } A = 60° \text{ and angle } ABC = 30°.$$

Again take the smallest side as unity, i.e. let $b = 1$. This makes

$$c = AB = AD = 2AC = 2b = 2,$$

and $\quad a = \sqrt{c^2 - b^2} = \sqrt{4 - 1} = \sqrt{3}$. Therefore



$$\sin 60° = \frac{\sqrt{3}}{2}; \qquad \csc 60° = \frac{2}{\sqrt{3}};$$

$$\cos 60° = \frac{1}{2}; \qquad \sec 60° = 2;$$

$$\tan 60° = \sqrt{3}; \qquad \cot 60° = \frac{1}{\sqrt{3}}.$$

Similarly, from the same triangle,

$$\sin 30° = \frac{1}{2}; \qquad \csc 30° = 2;$$

$$\cos 30° = \frac{\sqrt{3}}{2}; \qquad \sec 30° = \frac{2}{\sqrt{3}};$$

$$\tan 30° = \frac{1}{\sqrt{3}}; \qquad \cot 30° = \sqrt{3}.$$

Writing the more important of these results in tabulated form,* we have

ANGLE	30°	45°	60°
sin	$\frac{1}{2} = .50$	$\frac{1}{\sqrt{2}} = .71+$	$\frac{\sqrt{3}}{2} = .86+$
cos	$\frac{\sqrt{3}}{2} = .86+$	$\frac{1}{\sqrt{2}} = .71+$	$\frac{1}{2} = .50$
tan	$\frac{1}{\sqrt{3}} = .57+$	1	$\sqrt{3} = 1.73+$

The cosecant, secant, and cotangent are easily remembered as being the reciprocals of the sine, cosine, and tangent respectively.

* To aid the memory we observe that the numbers in the first (or sine) row are respectively $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$; each divided by 2.

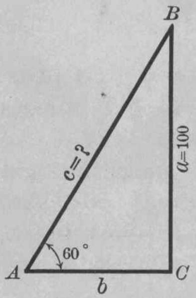
The second (or cosine) row is formed by reversing the order in the first row.

The last (or tangent) row is formed by dividing the numbers in the first row by the respective numbers in the second row.

The student should become very familiar with the 45° right triangle and the 30°, 60° right triangle. Instead of memorizing the above table we may then get the values of the functions directly from a mental picture of these right triangles.

Ex. 5. Given a right triangle where $A = 60°$, $a = 100$; find c.

Solution. Since we know A (and therefore also any function of A), and the sine of A involves a, which is known, and c, which is wanted, we can find c by using the formula



$$\sin A = \frac{a}{c}. \qquad \text{by (1), p. 2}$$

Substituting $a = 100$, and $\sin A = \sin 60° = \frac{\sqrt{3}}{2}$ from the above table, we have

$$\frac{\sqrt{3}}{2} = \frac{100}{c}.$$

Clearing of fractions and solving for c, we get

$$c = \frac{200}{\sqrt{3}} = \frac{200}{1.7+} = 117.6+. \quad \textit{Ans.}$$

What is the value of B? Following the method illustrated above, show that $b = 58.8+$.

EXAMPLES

Only right triangles are referred to in the following examples.

1. Calculate all the functions of the angle A, having given $a = 8$, $b = 15$.

Ans. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, etc.

2. Calculate the functions of the angle B, having given $a = 5$, $c = 7$.

Ans. $\sin B = \frac{\sqrt{24}}{7}$, $\cos B = \frac{5}{7}$, $\tan B = \frac{\sqrt{24}}{5}$, etc.

3. Calculate the functions of the angle A, having given $b = 2$, $c = \sqrt{11}$.

Ans. $\sqrt{\frac{7}{11}}$, $\frac{2}{\sqrt{11}}$, $\frac{\sqrt{7}}{2}$, etc.

4. Calculate the functions of the angle B, having given $a = 40$, $c = 41$.

Ans. $\frac{9}{41}$, $\frac{40}{41}$, $\frac{9}{40}$, etc.

5. Calculate the functions of the angle A, having given $a = p$, $b = q$.

Ans. $\frac{p}{\sqrt{p^2+q^2}}$, $\frac{q}{\sqrt{p^2+q^2}}$, $\frac{p}{q}$, etc.

6. Calculate the functions of the angle A, having given $a = \sqrt{m^2 + mn}$, $c = m + n$.

Ans. $\frac{\sqrt{m^2+mn}}{m+n}$, $\frac{\sqrt{mn+n^2}}{m+n}$, $\sqrt{\frac{m}{n}}$, etc.

7. Calculate the functions of the angle B, having given $a = \sqrt{m^2 + n^2}$, $c = m + n$.

Ans. $\frac{\sqrt{2mn}}{m+n}$, $\frac{\sqrt{m^2+n^2}}{m+n}$, $\sqrt{\frac{2mn}{m^2+n^2}}$, etc.

8. Given $\sin A = \frac{3}{5}$, $c = 200.5$; calculate a. *Ans.* 120.3.

9. Given $\cos A = .44$, $c = 30.5$; calculate b. *Ans.* 13.42.

10. Given $\tan A = \frac{11}{3}$, $b = \frac{27}{11}$; calculate c. *Ans.* $\frac{9}{11}\sqrt{130}$.

11. Given $A = 30°$, $a = 25$; calculate c. Also find B and b.
Ans. $c = 50$, $B = 60°$, $b = 25\sqrt{3}$.

12. Given $B = 30°$, $c = 48$; calculate b. Also find A and a.
Ans. $b = 24$, $A = 60°$, $a = 24\sqrt{3}$.

13. Given $B = 45°$, $b = 20$; calculate c. Also find A and a.
Ans. $c = 20\sqrt{2}$, $A = 45°$, $a = 20$.

3. Solution of right triangles. A triangle is composed of six parts, three sides and three angles. To solve a triangle is to find the parts not given. A triangle can be solved if three parts, at least one of which is a side, are given.* A right triangle has one angle, the right angle, always given; hence a right triangle can be solved if two sides, or one side and an acute angle, are given. One of the most important applications of Trigonometry † is the solution of triangles, and we shall now take up the *solution of right triangles.*

The student may have noticed that Examples 11, 12, 13, of the last section were really problems on solving right triangles.

When beginning the study of Trigonometry it is important that the student should draw the figures connected with the problems as accurately as possible. This not only leads to a better understanding of the problems themselves, but also gives a clearer insight into the meaning of the trigonometric functions and makes it possible to test roughly the accuracy of the results obtained. For this purpose the only instruments necessary are a graduated ruler and a protractor. A protractor is an instrument for measuring angles. On the inside of the back cover of this book will be found a Granville's Transparent Combined Ruler and Protractor, with directions for use. The ruler is graduated to inches and centimeters and the protractor to degrees. The student is advised to make free use of this instrument.

4. General directions for solving right triangles.

First step. *Draw a figure as accurately as possible representing the triangle in question.*

Second step. *When one acute angle is known, subtract it from 90° to get the other acute angle.*

* It is assumed that the given conditions are consistent, that is, that it is possible to construct the triangle from the given parts.

† The name Trigonometry is derived from two Greek words which taken together mean "I measure a triangle."