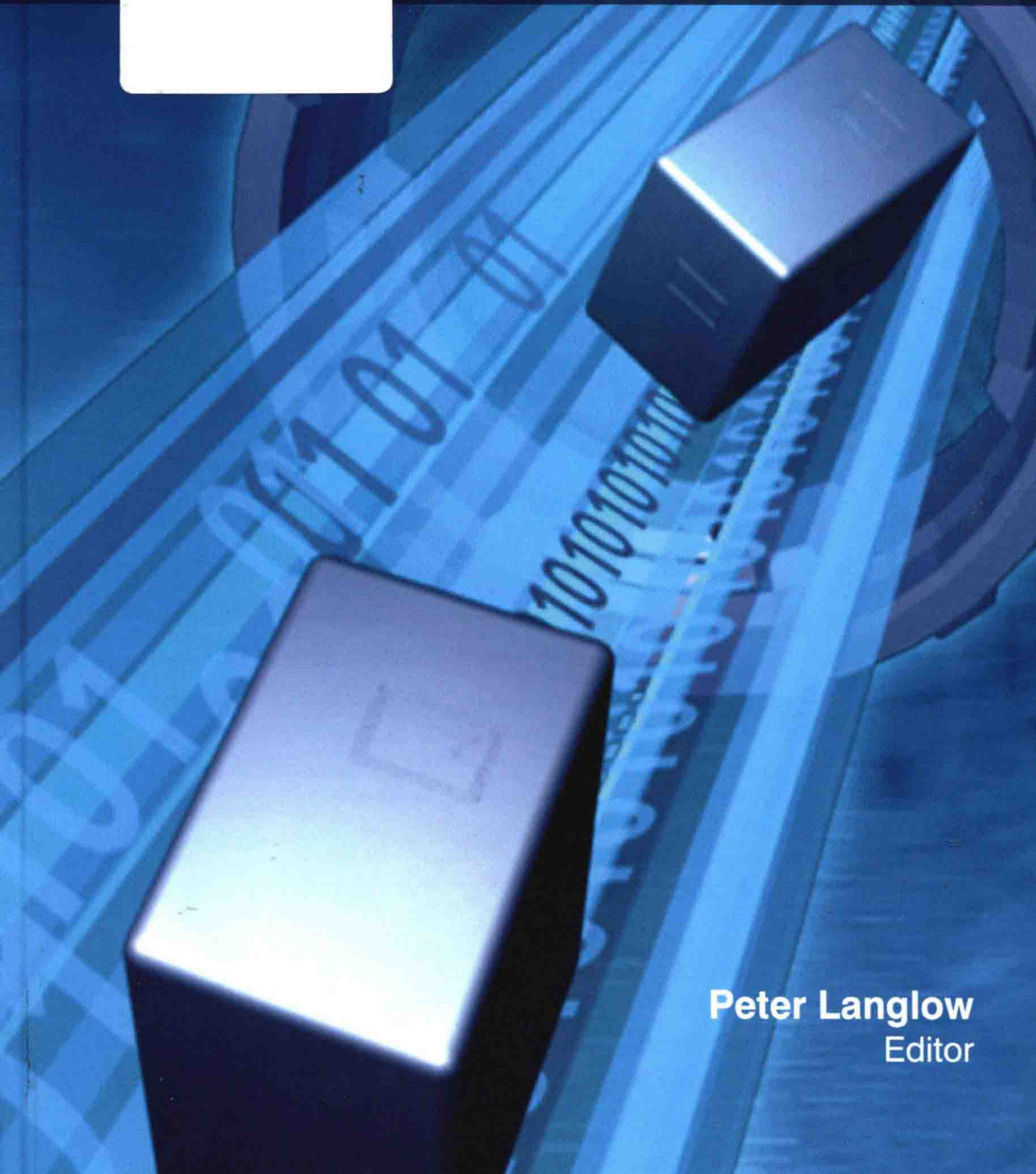
Handbook of

Optimization in Operations Research

Methods, Algorithms and Techniques



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Editor

Handbook of OPTIMIZATION IN OPERATIONS RESEARCH

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Preface

Operations research or operational research is a discipline that deals with the application of advanced analytical methods to help make better decisions. As a formal discipline, operational research originated in the efforts of military planners during World War II. In the decades after the war, the techniques began to be applied more widely to problems in business, industry and society. Since that time, operational research has expanded into a field widely used in industries ranging from petrochemicals to airlines, finance, logistics, and government, moving to a focus on the development of mathematical models that can be used to analyse and optimize complex systems, and has become an area of active academic and industrial research.

Operations research encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency, such as simulation, mathematical optimization, queuing theory and other stochastic-process models, Markov decision processes, econometric methods, data envelopment analysis, neural networks, expert systems, decision analysis, and the analytic hierarchy process. Nearly all of these techniques involve the construction of mathematical models that attempt to describe the system. Employing techniques from other mathematical sciences, such as mathematical modelling, statistical analysis, and mathematical optimization, operations research arrives at optimal or near-optimal solutions to complex decision-making problems. Because of its emphasis on human-technology interaction and because of its focus on practical applications, operations research has overlap with other disciplines, notably industrial engineering and operations management, and draws on psychology and organization science. Operations research is often concerned with determining the maximum or minimum of some real-world objective.

Optimization is a branch of operations research which uses mathematical techniques such as linear and nonlinear programming to derive values for system variables that will optimize performance. The mathematical aspects of operations research and systems analysis concerned with optimization of objectives form the subject of this book. The book discusses on linear programming and provides greater emphasis on duality theory, sensitivity analysis, parametric programming, multi-objective and goal programming and formulation and solution of practical problems. Aspects of Nonlinear programming including integer programming, kuhn-tucker theory, separable and quadratic programming, dynamic programming, geometric programming and direct search and gradient methods, theory of games and Karmarkar's projective algorithm is dealt with. The optimization techniques are needed in arriving at an effective decision in areas of management, finance, production, marketing, personnel, transportation, public health, military, planning and agriculture, etc. The persons connected with management and business executives, are therefore, supposed to have a good knowledge of optimization theory.

The book keeps in view the needs of the student taking a regular course in operations research or mathematical programming, and also of research scholars in other disciplines who have a limited objective of learning the practical aspects of various optimization methods to solve their special problems. Summaries of computational algorithms for various methods which would help him to write computer programmes to solve larger problems would be helpful to various sections. This book exposes students to the broad scope of the topic, reinforces the basic principles, sparks students' enthusiasm about the field, provides tools of immediate relevance and develops the skills necessary to use those tools.

—*Editor*

Contents

Chapter 1

Mathematical Optimization

In mathematics, computer science, or management science, mathematical optimization (alternatively, optimization or mathematical programming) is the selection of a best element (with regard to some criteria) from some set of available alternatives. In the simplest case, an optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function. The generalization of optimization theory and techniques to other formulations comprises a large area of applied mathematics. More generally, optimization includes finding "best available" values of some objective function given a defined domain, including a variety of different types of objective functions and different types of domains.

Optimization Problem

In mathematics and computer science, an optimization problem is the problem of finding the *best* solution from all feasible solutions. Optimization problems can be divided into two categories depending on whether the variables are continuous or discrete. An optimization problem with discrete variables is known as a combinatorial optimization problem. In a combinatorial optimization problem, we are looking for an object such as an integer, permutation or graph from a finite (or possibly countable infinite) set.

Continuous Optimization Problem

The *standard form* of a (continuous) optimization problem is

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1,\ldots,m \\ & h_i(x) = 0, \quad i = 1,\ldots,p \end{aligned}$$

where

- $f(x): \mathbb{R}^n \to \mathbb{R}$ is the objective function to be minimized over the variable x,
- $g_i(x) \leq 0$ are called inequality constraints, and
- $h_i(x) = 0$ are called equality constraints.

By convention, the standard form defines a minimization problem. A maximization problem can be treated by negating the objective function.

Combinatorial Optimization Problem

Formally, a combinatorial optimization problem A is a quadruple (I, f, m, g), where

- I is a set of instances;
- given an instance $x \in I$, $f(x)$ is the set of feasible solutions;
- given an instance x and a feasible solution y of $m(x, y)$, denotes the measure of y, which is usually a positive real.
- g is the goal function, and is either $\min$ or $\max$.

The goal is then to find for some instance an *optimal solution*, that is, a feasible solution y with

$$m(x,y) = g\{m(x,y') \mid y' \in f(x)\}.$$

For each combinatorial optimization problem, there is a corresponding decision problem that asks whether there is a feasible solution for some particular measure m_0. For example, if there is a graph G which contains vertices u and v, an optimization problem might be "find a path from to that uses the fewest edges". This problem might have an answer of, say, 4. A corresponding decision problem would be "is there a path from to that uses 10 or fewer edges?" This problem can be answered with a simple 'yes' or 'no'.

In the field of approximation algorithms, algorithms are designed to find near-optimal solutions to hard problems. The usual decision version is then an inadequate definition of the problem since it only specifies acceptable solutions. Even though we could introduce suitable decision problems, the problem is more naturally characterized as an optimization problem.

NP Optimization Problem

An *NP-optimization problem* (NPO) is a combinatorial optimization problem with the following additional conditions. Note that the below referred polynomials are functions of the size of the respective functions' inputs, not the size of some implicit set of input instances.

- the size of every feasible solution $y \in f(x)$ is polynomially bounded in the size of the given instance x,
- the languages $\{x|x \in I\}$ and $\{(x,y)|y \in f(x)\}$ can be recognised in polynomial time, and
- m is polynomial-time computable.

This implies that the corresponding decision problem is in NP. In computer science, interesting optimization problems usually have the above properties and are therefore NPO problems. A problem is additionally called a P-optimization (PO) problem, if there exists an algorithm which finds optimal solutions in polynomial time. Often, when dealing with the class NPO, one is interested in optimization problems for which the decision versions are NP-hard. Note that hardness relations are always with respect to some reduction. Due to the connection between approximation algorithms and computational optimization problems, reductions which preserve approximation in some respect are for this subject preferred than the usual Turing and Karp reductions. An example of such a reduction would be the L-reduction. For this reason, optimization problems with NP-complete decision versions are not necessarily called NPO-complete.

NPO is divided into the following subclasses according to their approximability:

- *NPO(I)*: Equals FPTAS. Contains the Knapsack problem.
- *NPO(II)*: Equals PTAS. Contains the Makespan scheduling problem.
- *NPO(III)*: :The class of NPO problems that have polynomial-time algorithms which computes solutions with a cost at most c times the optimal cost (for minimization problems) or a cost at least $1/c$ of the optimal cost (for maximization problems). In Hromkoviè's book, excluded from this class are all NPO(II)-problems save if P=NP. Without the exclusion, equals APX. Contains MAX-SAT and metric TSP.
- *NPO(IV)*: :The class of NPO problems with polynomial-time algorithms approximating the optimal solution by a ratio that is polynomial in a logarithm of the size of the input. In Hromkovic's book, all NPO(III)-problems are excluded from this class unless P=NP. Contains the set cover problem.
- *NPO(V)*: :The class of NPO problems with polynomial-time algorithms approximating the optimal solution by a ratio

bounded by some function on n. In Hromkovic's book, all NPO(IV)-problems are excluded from this class unless P=NP. Contains the TSP and Max Clique problems.

Another class of interest is NPOPB, NPO with polynomially bounded cost functions. Problems with this condition have many desirable properties.

Notation

Optimization problems are often expressed with special notation. Here are some examples.

Minimum and Maximum Value of a Function

Consider the following notation:

$$\min_{x\in\mathbb{R}} (x^2+1)$$

This denotes the minimum value of the objective function x^2+1, when choosing x from the set of real numbers $\mathbb{R}$. The minimum value in this case is 1, occurring at $x=0$.

Similarly, the notation

$$\max_{x\in\mathbb{R}} 2x$$

asks for the maximum value of the objective function $2x$, where x may be any real number. In this case, there is no such maximum as the objective function is unbounded, so the answer is "infinity" or "undefined".

Optimal Input Arguments

Consider the following notation:

$$\underset{x\in(-\infty,-1]}{\operatorname{argmin}}\, x^2+1,$$

or equivalently

$$\underset{x}{\operatorname{argmin}}\, x^2+1, \text{ subject to: } x\in(-\infty,-1].$$

This represents the value (or values) of the argument x in the interval $(-\infty,-1]$ that minimizes (or minimize) the objective function x^2+1 (the actual minimum value of that function is not what the problem asks for). In this case, the answer is $x = -1$, since $x = 0$ is infeasible, i.e. does not belong to the feasible set.

Similarly,

$$\underset{x\in[-5,5],\, y\in\mathbb{R}}{\operatorname{argmax}}\, x\cos(y),$$

or equivalently

$$\underset{x,y}{\arg\max}\; x\cos(y), \text{ subject to: } x \in [-5,5],\; y \in \mathbb{R},$$

represents the (x,y) pair (or pairs) that maximizes (or maximize) the value of the objective function $x\cos(y)$, with the added constraint that x lie in the interval [–5,5] (again, the actual maximum value of the expression does not matter). In this case, the solutions are the pairs of the form (5, 2kπ) and (–5,(2k+1)π), where k ranges over all integers.

Arg min and arg max are sometimes also written argmin and argmax, and stand for argument of the minimum and argument of the maximum.

History

Fermat and Lagrange found calculus-based formulas for identifying optima, while Newton and Gauss proposed iterative methods for moving towards an optimum. Historically, the first term for optimization was "linear programming", which was due to George B. Dantzig, although much of the theory had been introduced by Leonid Kantorovich in 1939. Dantzig published the Simplex algorithm in 1947, and John von Neumann developed the theory of duality in the same year.

The term, *programming*, in this context does not refer to computer programming. Rather, the term comes from the use of *program* by the United States military to refer to proposed training and logistics schedules, which were the problems Dantzig studied at that time.

Multi-Objective Optimization

Multi-objective optimization (also known as multi-objective programming, vector optimization, multicriteria optimization, multiattribute optimization or Pareto optimization) is an area of multiple criteria decision making, that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously. Multi-objective optimization has been applied in many fields of science, including engineering, economics and logistics, where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. Minimizing weight while maximizing the strength of a particular component, and maximizing performance whilst minimizing fuel consumption and emission of pollutants of a vehicle are examples of multi-objective optimization problems involving two and three objectives, respectively. In practical problems, there can be more than three objectives.

For a nontrivial multi-objective optimization problem, there does not exist a single solution that simultaneously optimizes each objective. In that case, the objective functions are said to be conflicting, and there exists a (possibly infinite number of) Pareto optimal solutions. A solution is called nondominated, Pareto optimal, Pareto efficient or noninferior, if none of the objective functions can be improved in value without degrading some of the other objective values. Without additional subjective preference information, all Pareto optimal solutions are considered equally good (as vectors cannot be ordered completely). Researchers study multi-objective optimization problems from different viewpoints and, thus, there exist different solution philosophies and goals when setting and solving them. The goal may be to find a representative set of Pareto optimal solutions, and/or quantify the trade-offs in satisfying the different objectives, and/or finding a single solution that satisfies the subjective preferences of a human decision maker (DM).

Introduction

A multi-objective optimization problem is an optimization problem that involves multiple objective functions. In mathematical terms, a multi-objective optimization problem can be formulated as

$$\begin{aligned} \min \quad & \left(f_1(x), f_2(x), \ldots, f_k(x)\right) \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where the integer $k \geq 2$ is the number of objectives and the set X is the feasible set of decision vectors. The feasible set is typically defined by some constraint functions. In addition, the vector-valued objective function is often defined as

$$f : X \to \mathbb{R}^k,\ f(x) = (f_1(x), \ldots, f_k(x))^T .$$

If some objective function is to be maximized, it is equivalent to minimize its negative. The image of X is denoted by $Y \in \mathbb{R}^k$

An element $x^* \in X$ is called a feasible solution or a feasible decision. A vector $z^* := f(x^*) \in \mathbb{R}^k$ for a feasible solution x^* is called an objective vector or an outcome. In multi-objective optimization, there does not typically exist a feasible solution that minimizes all objective functions simultaneously. Therefore, attention is paid to Pareto optimal solutions; that is, solutions that cannot be improved in any of the objectives without degrading at least one of the other objectives. In mathematical terms, a feasible solution $x^1 \in X$ is said to (Pareto) dominate another solution $x^2 \in X$, if

1. $f_i(x^1) \le f_i(x^2)$ for all indices $i \in \{1,2,\ldots,k\}$ and
2. $f_j(x^1) < f_j(x^2)$ for at least one index $j \in \{1,2,\ldots,k\}$.

A solution $x^1 \in X$ (and the corresponding outcome $f(x^*)$) is called Pareto optimal, if there does not exist another solution that dominates it. The set of Pareto optimal outcomes is often called the Pareto front or Pareto boundary.

The Pareto front of a multi-objective optimization problem is bounded by a so-called nadir objective vector z^{nad} and an ideal objective vector z^{ideal}, if these are finite. The nadir objective vector is defined as

$$z_i^{nad} = \sup_{x \in X \text{ is Pareto optimal}} f_i(x) \text{ for all } i = 1,\ldots,k$$

and the ideal objective vector as

$$z_i^{ideal} = \inf f_i(x) \text{ for all } i = 1,\ldots,k.$$

In other words, the components of a nadir and an ideal objective vector define upper and lower bounds for the objective function values of Pareto optimal solutions, respectively. In practice, the nadir objective vector can only be approximated as, typically, the whole Pareto optimal set is unknown.

Examples of Multi-Objective Optimization Applications

Economics

In economics, many problems involve multiple objectives along with constraints on what combinations of those objectives are attainable. For example, consumer's demand for various goods is determined by the process of maximization of the utilities derived from those goods, subject to a constraint based on how much income is available to spend on those goods and on the prices of those goods. This constraint allows more of one good to be purchased only at the sacrifice of consuming less of another good; therefore, the various objectives (more consumption of each good is preferred) are in conflict with each other. A common method for analyzing such a problem is to use a graph of indifference curves, representing preferences, and a budget constraint, representing the trade-offs that the consumer is faced with.

Another example involves the production possibilities frontier, which specifies what combinations of various types of goods can be produced by a society with certain amounts of various resources. The frontier specifies the trade-offs that the society is faced with — if the society is fully utilising its resources, more of one good can be produced only at the expense of producing less of another good. A society must then use some process to choose among the possibilities on the frontier.

Macroeconomic policy-making is a context requiring multi-objective optimization. Typically a central bank must choose a stance for monetary policy that balances competing objectives — low inflation, low unemployment, low balance of trade deficit, etc. To do this, the central bank uses a model of the economy that quantitatively describes the various causal linkages in the economy; it simulates the model repeatedly under various possible stances of monetary policy, in order to obtain a menu of possible predicted outcomes for the various variables of interest. Then in principle it can use an aggregate objective function to rate the alternative sets of predicted outcomes, although in practice central banks use a non-quantitative, judgement-based, process for ranking the alternatives and making the policy choice.

Finance

In finance, a common problem is to choose a portfolio when there are two conflicting objectives — the desire to have the expected value of portfolio returns be as high as possible, and the desire to have risk, measured by the standard deviation of portfolio returns, be as low as possible. This problem is often represented by a graph in which the efficient frontier shows the best combinations of risk and expected return that are available, and in which indifference curves show the investor's preferences for various risk-expected return combinations. The problem of optimizing a function of the expected value (first moment) and the standard deviation (square root of the second moment) of portfolio return is called a two-moment decision model.

Optimal Control

In engineering and economics, many problems involve multiple objectives which are not describable as the-more-the-better or the-less-the-better; instead, there is an ideal target value for each objective, and the desire is to get as close as possible to the desired value of each objective. For example, one might want to adjust a rocket's fuel usage and orientation so that it arrives both at a specified place and at a specified time; or one might want to conduct open market operations so that both the inflation rate and the unemployment rate are as close as possible to their desired values.

Often such problems are subject to linear equality constraints that prevent all objectives from being simultaneously perfectly met,