# EXAMPLES

OF

# DIFFERENTIAL EQUATIONS.

OSBORNE

# EXAMPLES

OF

# DIFFERENTIAL EQUATIONS

WITH

## RULES FOR THEIR SOLUTION.

BY

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# PREFACE.

THIS work has been prepared to meet a want felt by the author in a practical course on the subject, arranged for advanced students in Physics. It is intended to be used in connection with lectures on the theory of Differential Equations and the derivation of the methods of solution.

Many of the examples have been collected from standard treatises, but a considerable number have been prepared by the author to illustrate special difficulties, or to provide exercises corresponding more nearly with the abilities of average students. With few exceptions they have all been tested by use in the class-room.

G. A. OSBORNE.

BOSTON, Feb. 1, 1886.

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# CONTENTS.

## CHAPTER I.

DEFINITIONS. — DERIVATION OF THE DIFFERENTIAL EQUATION FROM THE COMPLETE PRIMITIVE.



## *SOLUTION OF DIFFERENTIAL EQUATIONS.*

## CHAPTER II.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND DEGREE BETWEEN TWO VARIABLES.



## CHAPTER III.

EXACT DIFFERENTIAL EQUATIONS. — INTEGRATING FACTORS.

## CHAPTER IV.

### DIFFERENTIAL EQUATIONS OF FIRST ORDER AND DEGREE WITH THREE VARIABLES.



## CHAPTER V.

### DIFFERENTIAL EQUATIONS OF THE FIRST ORDER OF HIGHER DEGREES.



## CHAPTER VI.

### SINGULAR SOLUTIONS.



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# *DIFFERENTIAL EQUATIONS OF HIGHER ORDERS.*

## CHAPTER VII.

### LINEAR DIFFERENTIAL EQUATIONS.

## CHAPTER VIII.

### SPECIAL FORMS OF DIFFERENTIAL EQUATIONS OF HIGHER ORDERS.



## CHAPTER IX.

### SIMULTANEOUS DIFFERENTIAL EQUATIONS.



## CHAPTER X.

# EXAMPLES OF DIFFERENTIAL EQUATIONS.

## CHAPTER I.

### DEFINITIONS. DERIVATION OF THE DIFFERENTIAL EQUATION FROM THE COMPLETE PRIMITIVE.

**1.** A *differential equation* is an equation containing differentials or differential coefficients.

The *solution* of a differential equation is the determination of another equation free from differentials or differential coefficients, from which the former may be derived by differentiation.

The *order* of a differential equation is that of the highest differential coefficient it contains; and its *degree* is that of the highest power to which this highest differential coefficient is raised, after the equation is freed from fractions and radicals.

The solution of a differential equation requires one or more integrations, each of which introduces an arbitrary constant. The most general solution of a differential equation of the $n$th order contains $n$ arbitrary constants, whatever may be its degree. This general solution is called the *complete primitive* of the given differential equation.

**2.** *To derive a differential equation of the first order from its complete primitive.*

Differentiate the primitive; and if the arbitrary constant has disappeared, the result is the required differential equation. If not, the elimination of this constant between the two equations will give the differential equation.

**3.** Form the differential equations of the first order of which the following are the complete primitives, $c$ being the arbitrary constant:

1. $\log(xy) + x = y + c.$
2. $(1 + x^2)(1 + y^2) = cx^2.$
3. $\cos y = c \cos x.$
4. $y = ce^{-\tan^{-1} x} + \tan^{-1} x - 1.$
5. $y = (cx + \log x + 1)^{-1}.$
6. $y = cx + c - c^3.$
7. $(y + c)^2 = 4ax.$
8. $y^2 \sin^2 x + 2cy + c^2 = 0.$
9. $e^{2y} + 2cxe^y + c^2 = 0.$

**4.** *To derive a differential equation of the second order from its complete primitive.*

Differentiate the primitive twice successively, and eliminate, if necessary, the two arbitrary constants between the three equations.

**5.** Form the differential equations of the second order of which the following are the complete primitives, $c_1$ and $c_2$ being the arbitrary constants:

1. $y = c_1 \cos(ax + c_2).$
2. $y = c_1 e^{ax} + c_2 e^{-ax}.$
3. $y = (c_1 + c_2 x)e^{ax}.$
4. $y = c_1 x^3 + \frac{c_2}{x}.$
5. $y = c_1 \sin nx + c_2 \cos nx + \frac{\cos ax}{n^2 - a^2}.$

**6.** The preceding process may be extended to the derivation of equations of higher orders from their primitives.

**7.** Form the differential equations of the third order of which the following are the complete primitives:

1. $y = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^x.$

2. $ye^x = c_1 e^{2x} + c_2 \sin x\sqrt{2} + c_3 \cos x\sqrt{2}.$

3. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right)e^x + c_3.$

Form the differential equations of the fourth order of which the following are the complete primitives:

4. $y = (c_1 + c_2 x + c_3 x^2)e^x + c_4.$

5. $x^3 + a^4 y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \sin ax + c_4 \cos ax.$

## CHAPTER II.

### DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE BETWEEN TWO VARIABLES.

General Form, $Mdx + Ndy = 0$,

where $M$, $N$, are each functions of $x$ and $y$.

**8**. Form, $XYdx + X'Y'dy = 0$,

where $X$, $X'$, are functions of $x$ alone, and $Y$, $Y'$, functions of $y$ alone.

Divide so as to separate the variables, and integrate each part separately.

**9**. Solve the following equations:

**1.** $(1+x)y\,dx + (1-y)x\,dy = 0.$

**2.** $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0.$

**3.** $\dfrac{dy}{dx} = \dfrac{1+y^2}{(1+x^2)xy}.$

**4.** $a\left(x\dfrac{dy}{dx} + 2y\right) = xy\dfrac{dy}{dx}.$

**5.** $(1+y^2)dx = (y + \sqrt{1+y^2})(1+x^2)^{\frac{3}{2}}dy.$

**6.** $\sin x \cos y\, dx = \cos x \sin y\, dy.$

**7.** $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$

**8.** $\sec^2 x \tan y\, dy + \sec^2 y \tan x\, dx = 0.$

**9.** $\dfrac{dy}{dx} + \dfrac{1+y+y^2}{1+x+x^2} = 0.$

**10.** *Homogeneous equations.*

Substitute $y = vx$; in the resulting equation between $v$ and $x$, the variables can be separated. (See Art. 8.)

**11.** Solve the following equations:

1. $(y - x)\,dy + y\,dx = 0.$

2. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$

3. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$

4. $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}.$

5. $x\cos\dfrac{y}{x}\cdot\dfrac{dy}{dx} = y\cos\dfrac{y}{x} - x.$

6. $(8y + 10x)\,dx + (5y + 7x)\,dy = 0.$

7. $(x + y)\dfrac{dy}{dx} = y - x.$

8. $x\cos\dfrac{y}{x}(y\,dx + x\,dy) = y\sin\dfrac{y}{x}(x\,dy - y\,dx).$

9. $x + y\dfrac{dy}{dx} = my.$

   (1), $m < 2$; (2), $m = 2$; (3), $m > 2$.

10. $$[(x^2 - y^2)\sin\alpha + 2xy\cos\alpha - y\sqrt{x^2 + y^2}]\frac{dy}{dx} = 2xy\sin\alpha - (x^2 - y^2)\cos\alpha + x\sqrt{x^2 + y^2}.$$

**12.** Form,

$$(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0.$$

Substitute $x = x' + \alpha,\quad y = y' + \beta,$

and determine the constants $\alpha$, $\beta$, so that the new equation between $x'$ and $y'$ may be homogeneous. (See Art. 10.)