Topics in Matrix Analysis

矩阵分析

（英文版）

［美］ Roger A. Horn
Charles R. Johnson 著

TOPICS IN
MATRIX ANALYSIS

ROGER A. HORN
AND
CHARLES R. JOHNSON

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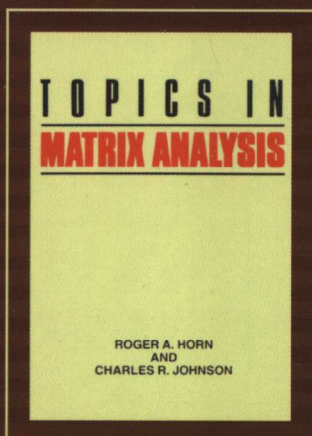
Topics in Matrix Analysis
矩阵分析 卷2（英文版）

"Horn 和 Johnson 教授合著的《矩阵分析 卷1》和《矩阵分析 卷2》是不可或缺的。首先，深入地探讨了每个论题；其次，书中涵盖了一些高级论题；第三，精心组织，内容广泛，是一本标准的教科书。"

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"毫无疑问，《矩阵分析 卷1》和《矩阵分析 卷2》是当今矩阵理论文献中非常重要的而且无可比拟的著作。"

——美国数学学会会刊

本书是继《矩阵分析 卷1》之后推出的矩阵领域又一经典之作，详尽讨论了卷1未能包括但又具有极高应用价值的论题。

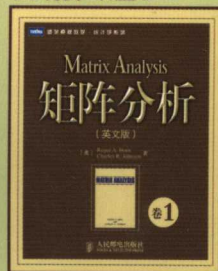书中包含大量矩阵理论和线性代数方面的经典定理和推论，并给出了严格的证明。很多定理、推论、论题等都是本书所独有的，再加上作者精心组织，言简意赅的表述，造就了本书在矩阵领域不可比拟、独一无二的地位。

Roger A. Horn 线性代数和矩阵理论领域国际知名权威。1967年获得斯坦福大学数学博士，1972~1979年任约翰·霍普金斯大学数学系系主任，现为犹他大学教授。曾担任 *American Mathematical Monthly* 编辑。

Charles R. Johnson 线性代数和矩阵理论领域国际知名权威。现为威廉玛丽学院教授。Johnson 在学术界十分活跃，发表论文近300篇，担任过多个主要矩阵分析类杂志的编辑和两份 SIAM 杂志的主编。由于他在数学科学领域作出了杰出贡献而被授予华盛顿科学学会奖。

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Topics in Matrix Analysis

矩阵分析 卷 2

（英文版）

[美] Roger A. Horn
Charles R. Johnson 著

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内 容 提 要

　　本书在《矩阵分析 卷 1》基础之上，详尽叙述了卷 1 未能包括的但又具有极高应用价值的论题。这些论题包括：值域、稳定矩阵和惯性、奇异值、矩阵方程和 Kronecker 乘积、Hadamard 乘积、矩阵和函数。

　　本书可作为数学及工程领域的研究生和研究人员的深入学习矩阵理论的教科书或参考书。

Preface

This volume is a sequel to the previously published *Matrix Analysis* and includes development of further topics that support applications of matrix theory. We refer the reader to the preface of the prior volume for many general comments that apply here also. We adopt the notation and referencing conventions of that volume and make specific reference to it [HJ] as needed.

Matrix Analysis developed the topics of broadest utility in the connection of matrix theory to other subjects and for modern research in the subject. The current volume develops a further set of slightly more specialized topics in the same spirit. These are: the field of values (or classical numerical range), matrix stability and inertia (including *M*-matrices), singular values and associated inequalities, matrix equations and Kronecker products, Hadamard (or entrywise) products of matrices, and several ways in which matrices and functions interact. Each of these topics is an area of active current research, and several of them do not yet enjoy a broad exposition elsewhere.

Though this book should serve as a reference for these topics, the exposition is designed for use in an advanced course. Chapters include motivational background, discussion, relations to other topics, and literature references. Most sections include exercises in the development as well as many problems that reinforce or extend the subject under discussion. There are, of course, other matrix analysis topics not developed here that warrant attention. Some of these already enjoy useful expositions; for example, totally positive matrices are discussed in [And] and [Kar].

We have included many exercises and over 650 problems because we feel they are essential to the development of an understanding of the subject and its implications. The exercises occur throughout the text as part of the

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development of each section; they are generally elementary and of immediate use in understanding the concepts. We recommend that the reader work at least a broad selection of these. Problems are listed (in no particular order) at the end of sections; they cover a range of difficulties and types (from theoretical to computational) and they may extend the topic, develop special aspects, or suggest alternate proofs of major ideas. In order to enhance the utility of the book as a reference, many problems have hints; these are collected in a separate section following Chapter 6. The results of some problems are referred to in other problems or in the text itself. We cannot overemphasize the importance of the reader's active involvement in carrying out the exercises and solving problems.

As in the prior volume, a broad list of related books and major surveys is given prior to the index, and references to this list are given via mnemonic code in square brackets. Readers may find the reference list of independent utility.

We appreciate the assistance of our colleagues and students who have offered helpful suggestions or commented on the manuscripts that preceded publication of this volume. They include M. Bakonyi, W. Barrett, O. Chan, C. Cullen, M. Cusick, J. Dietrich, S. H. Friedberg, S. Gabriel, F. Hall, C.-K. Li, M. Lundquist, R. Mathias, D. Merino, R. Merris, P. Nylen, A. Sourour, G. W. Stewart, R. C. Thompson, P. van Dooren, and E. M. E. Wermuth.

The authors wish to maintain the utility of this volume to the community and welcome communication from readers of errors or omissions that they find. Such communications will be rewarded with a current copy of all known errata.

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R. A. H.
C. R. J.

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Preface to the Second Printing

We have corrected all known errata in the first printing, polished the exposition of a few points, noted the resolution of several conjectures, and added some items to the notation list and index. It is a pleasure to acknowledge helpful comments from our colleagues T. Ando, R. Bhatia, S. Friedberg, D. Jesperson, B. Kroschel, I. Lewkowicz, C.-K. Li, R. Loewy, J. Miao, and F. Uhlig.

Contents

Chapter 1

The field of values

1.0   Introduction

Like the spectrum (or set of eigenvalues) $\sigma(\cdot)$, the field of values $F(\cdot)$ is a
set of complex numbers naturally associated with a given n-by-n matrix A:

$$F(A) \equiv \{x^* A x \colon\ x \in \mathbb{C}^n,\ x^* x = 1\}$$

The spectrum of a matrix is a discrete point set; while the field of values can
be a continuum, it is always a compact convex set. Like the spectrum, the
field of values is a set that can be used to learn something about the matrix,
and it can often give information that the spectrum alone cannot give. The
eigenvalues of Hermitian and normal matrices have especially pleasant
properties, and the field of values captures certain aspects of this nice struc-
ture for general matrices.

1.0.1   Subadditivity and eigenvalues of sums

If only the eigenvalues $\sigma(A)$ and $\sigma(B)$ are known about two n-by-n matrices
A and B, remarkably little can be said about $\sigma(A + B)$, the eigenvalues of
the sum. Of course, $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$, so the sum of all the eigen-
values of $A + B$ is the sum of all the eigenvalues of A plus the sum of all the
eigenvalues of B. But beyond this, nothing can be said about the eigenvalues
of $A + B$ without more information about A and B. For example, even if all
the eigenvalues of two n-by-n matrices A and B are known and fixed, the
spectral radius of $A + B$ [the largest absolute value of an eigenvalue of
$A + B$, denoted by $\rho(A + B)$] can be arbitrarily large (see Problem 1). On
the other hand, if A and B are normal, then much can be said about the

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eigenvalues of $A + B$; for example, $\rho(A + B) \leq \rho(A) + \rho(B)$ in this case. Sums of matrices do arise in practice, and two relevant properties of the field of values $F(\cdot)$ are:

(a)   The field of values is subadditive: $F(A + B) \subset F(A) + F(B)$, where the set sum has the natural definition of sums of all possible pairs, one from each; and

(b)   The eigenvalues of a matrix lie inside its field of values: $\sigma(A) \subset F(A)$.

Combining these two properties yields the inclusions

$$\sigma(A + B) \subset F(A + B) \subset F(A) + F(B)$$

so if the two fields of values $F(A)$ and $F(B)$ are known, something can be said about the spectrum of the sum.

1.0.2   An application from the numerical solution of partial differential equations

Suppose that $A = [a_{ij}] \in M_n(\mathbb{R})$ satisfies

(a) A is tridiagonal ($a_{ij} = 0$ for $|i - j| > 1$), and

(b) $a_{i,i+1} a_{i+1,i} < 0$ for $i = 1, \ldots, n - 1$.

Matrices of this type arise in the numerical solution of partial differential equations and in the analysis of dynamical systems arising in mathematical biology. In both cases, knowledge about the real parts of the eigenvalues of A is important. It turns out that rather good information about the eigenvalues of such a matrix can be obtained easily using the field of values $F(\cdot)$.

1.0.2.1   Fact: For any eigenvalue λ of a matrix A of the type indicated, we have

$$\min_{1 \leq i \leq n} a_{ii} \leq \operatorname{Re} \lambda \leq \max_{1 \leq i \leq n} a_{ii}$$

A proof of this fact is fairly simple using some properties of the field of values to be developed in Section (1.2). First, choose a diagonal matrix D

with positive diagonal entries such that $D^{-1}AD \equiv \hat{A} = [\hat{a}_{ij}]$ satisfies $\hat{a}_{ji} = -\hat{a}_{ij}$ for $j \neq i$. The matrix $D \equiv \text{diag}(d_1, ..., d_n)$ defined by

$$d_1 = 1, \text{ and } d_i = \left| \frac{a_{i,i-1}}{a_{i-1,i}} \right|^{\frac{1}{2}} d_{i-1}, \; d_i > 0, \; i = 2, ..., n$$

will do. Since $\hat{A}$ and A are similar, their eigenvalues are the same. We then have

$$\text{Re } \sigma(A) = \text{Re } \sigma(\hat{A}) \subset \text{Re } F(\hat{A}) = F(\tfrac{1}{2}(\hat{A} + \hat{A}^T))$$
$$= F(\text{diag}(a_{11}, ..., a_{nn}))$$
$$= \text{Convex hull of } \{a_{11}, ..., a_{nn}\} = [\min_i a_{ii}, \max_i a_{ii}]$$

The first inclusion follows from the spectral containment property (1.2.6), the next equality follows from the projection property (1.2.5), the next equality follows from the special form achieved for $\hat{A}$, and the last equality follows from the normality property (1.2.9) and the fact that the eigenvalues of a diagonal matrix are its diagonal entries. Since the real part of each eigenvalue $\lambda \in \sigma(A)$ is a convex combination of the main diagonal entries a_{ii}, $i = 1, ..., n$, the asserted inequalities are clear and the proof is complete.

1.0.3   Stability analysis

In an analysis of the stability of an equilibrium in a dynamical system governed by a system of differential equations, it is important to know if the real part of every eigenvalue of a certain matrix A is negative. Such a matrix is called *stable*. In order to avoid juggling negative signs, we often work with *positive stable* matrices (all eigenvalues have positive real parts). Obviously, A is positive stable if and only if $-A$ is stable. An important sufficient condition for a matrix to be positive stable is the following fact.

1.0.3.1   Fact: Let $A \in M_n$. If $A + A^*$ is positive definite, then A is positive stable.

This is another application of properties of the field of values $F(\cdot)$ to be developed in Section (1.2). By the spectral containment property (1.2.6), $\text{Re } \sigma(A) \subset \text{Re } F(A)$, and, by the projection property (1.2.5), $\text{Re } F(A) =$

$F(\frac{1}{2}(A + A^*))$. But, since $A + A^*$ is positive definite, so is $\frac{1}{2}(A + A^*)$, and hence, by the normality property (1.2.9), $F(\frac{1}{2}(A + A^*))$ is contained in the positive real axis. Thus, each eigenvalue of A has a positive real part, and A is positive stable.

Actually, more is true. If $A + A^*$ is positive definite, and if $P \in M_n$ is any positive definite matrix, then PA is positive stable because

$$(P^{\frac{1}{2}})^{-1}[PA]P^{\frac{1}{2}} = P^{\frac{1}{2}}AP^{\frac{1}{2}}, \text{ and}$$

$$P^{\frac{1}{2}}AP^{\frac{1}{2}} + (P^{\frac{1}{2}}AP^{\frac{1}{2}})^* = P^{\frac{1}{2}}(A + A^*)P^{\frac{1}{2}}$$

where $P^{\frac{1}{2}}$ is the unique (Hermitian) positive definite square root of P. Since congruence preserves positive definiteness, the eigenvalues of PA have positive real parts for the same reason as A. Lyapunov's theorem (2.2.1) shows that all positive stable matrices arise in this way.

1.0.4   An approximation problem

Suppose we wish to approximate a given matrix $A \in M_n$ by a complex multiple of a Hermitian matrix of rank at most one, as closely as possible in the Frobenius norm $\|\cdot\|_2$. This is the problem

$$\text{minimize } \|A - cxx^*\|_2^2 \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1 \text{ and } c \in \mathbb{C} \qquad (1.0.4.1)$$

Since the inner product $[A,B] \equiv \text{tr } AB^*$ generates the Frobenius norm, we have

$$\|A - cxx^*\|_2^2 = [A - cxx^*, A - cxx^*]$$

$$= \|A\|_2^2 - 2 \text{ Re } \bar{c}[A,xx^*] + |c|^2$$

which, for a given unit vector x, is minimized by $c = [A,xx^*]$. Substitution of this value into (1.0.4.1) transforms our problem into

$$\text{minimize } (\|A\|_2^2 - |[A,xx^*]|^2) \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1$$

or, equivalently,

$$\text{maximize } |[A,xx^*]| \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1$$

A vector x_0 that solves the latter problem (we are maximizing a continuous function on a compact set) will yield a rank one solution $[A, x_0 x_0^*] x_0 x_0^*$ to our original problem. Since $[A, xx^*] = \operatorname{tr} A xx^* = x^* A x$, we are led naturally to finding a unit vector x such that the point $x^* A x$ in the field of values $F(A)$ has maximum distance from the origin. The absolute value of such a point is called the *numerical radius* of A [often denoted by $r(A)$] by analogy with the *spectral radius* [often denoted by $\rho(A)$], which is the absolute value of a point in the spectrum $\sigma(A)$ that is at maximum distance from the origin.

Problems

1.   Consider the real matrices

$$A = \begin{bmatrix} 1 - \alpha & 1 \\ \alpha(1 - \alpha) - 1 & \alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 + \alpha & 1 \\ -\alpha(1 + \alpha) - 1 & -\alpha \end{bmatrix}$$

Show that $\sigma(A)$ and $\sigma(B)$ are independent of the value of $\alpha \in \mathbb{R}$. What are they? What is $\sigma(A + B)$? Show that $\rho(A + B)$ is unbounded as $\alpha \to \infty$.

2.   In contrast to Problem 1, show that if $A, B \in M_n$ are normal, then $\rho(A + B) \leq \rho(A) + \rho(B)$.

3.   Show that "$<$" in (1.0.2(b)) may be replaced by "$\leq$," the main diagonal entries a_{ii} may be complex, and Fact (1.0.2.1) still holds if a_{ii} is replaced by $\operatorname{Re} a_{ii}$.

4.   Show that the problem of approximating a given $A \in M_n$ by a positive semidefinite rank one matrix with spectral radius one can be solved if one can find a unit vector x such that the point $x^* A x$ in $F(A)$ is furthest to the right in the complex plane, that is, $\operatorname{Re} x^* A x$ is maximized.

1.1   Definitions

In this section we define the field of values and certain related objects.

1.1.1   Definition. The *field of values* of $A \in M_n$ is

$$F(A) \equiv \{ x^* A x : x \in \mathbb{C}^n, x^* x = 1 \}$$

Thus, $F(\cdot)$ is a function from M_n into subsets of the complex plane.

$F(A)$ is just the normalized locus of the Hermitian form associated with A. The field of values is often called the *numerical range*, especially in the context of its analog for operators on infinite dimensional spaces.

Exercise. Show that $F(I) = \{1\}$ and $F(\alpha I) = \{\alpha\}$ for all $\alpha \in \mathbb{C}$. Show that $F\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the closed unit interval $[0,1]$, and $F\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is the closed unit disc $\{z \in \mathbb{C}: |z| \leq 1\}$.

The field of values $F(A)$ may also be thought of as the image of the surface of the Euclidean unit ball in $\mathbb{C}^n$ (a compact set) under the continuous transformation $x \to x^*Ax$. As such, $F(A)$ is a compact (and hence bounded) set in $\mathbb{C}$. An unbounded analog of $F(\cdot)$ is also of interest.

1.1.2   Definition. The *angular field of values* is

$$F'(A) \equiv \{x^*Ax: x \in \mathbb{C}^n, x \neq 0\}$$

Exercise. Show that $F'(A)$ is determined geometrically by $F(A)$; every open ray from the origin that intersects $F(A)$ in a point other than the origin is in $F'(A)$, and $0 \in F'(A)$ if and only if $0 \in F(A)$. Draw a typical picture of an $F(A)$ and $F'(A)$ assuming that $0 \notin F(A)$.

It will become clear that $F'(A)$ is an angular sector of the complex plane that is anchored at the origin (possibly the entire complex plane). The angular opening of this sector is of interest.

1.1.3   Definition. The *field angle* $\Theta \equiv \Theta(A) \equiv \Theta(F'(A)) \equiv \Theta(F(A))$ of $A \in M_n$ is defined as follows:

  (a)   If 0 is an interior point of $F(A)$, then $\Theta(A) \equiv 2\pi$.
  (b)   If 0 is on the boundary of $F(A)$ and there is a (unique) tangent to the boundary of $F(A)$ at 0, then $\Theta(A) \equiv \pi$.
  (c)   If $F(A)$ is contained in a line through the origin, $\Theta(A) \equiv 0$.
  (d)   Otherwise, consider the two different support lines of $F(A)$ that go through the origin, and let $\Theta(A)$ be the angle subtended by these two lines at the origin. If $0 \notin F(A)$, these support lines will be uniquely determined; if 0 is on the boundary of $F(A)$, choose the two support lines that give the minimum angle.

We shall see that $F(A)$ is a compact convex set for every $A \in M_n$, so this informal definition of the field angle makes sense. The field angle is just the angular opening of the smallest angular sector that includes $F(A)$, that is, the angular opening of the sector $F'(A)$.

Finally, the size of the bounded set $F(A)$ is of interest. We measure its size in terms of the radius of the smallest circle centered at the origin that contains $F(A)$.

1.1.4   Definition. The *numerical radius* of $A \in M_n$ is

$$r(A) \equiv \max \{|z| : z \in F(A)\}$$

The numerical radius is a vector norm on matrices that is not a matrix norm (see Section (5.7) of [HJ]).

Problems

1.   Show that among the vectors entering into the definition of $F(A)$, only vectors with real nonnegative first coordinate need be considered.

2.   Show that both $F(A)$ and $F'(A)$ are simply connected for any $A \in M_n$.

3.   Show that for each $0 \le \theta \le \pi$, there is an $A \in M_2$ with $\Theta(A) = \theta$. Is $\Theta(A) = 3\pi/2$ possible?

4.   Why is the "max" in $(1.1.4)$ attained?

5.   Show that the following alternative definition of $F(A)$ is equivalent to the one given:

$$F(A) \equiv \{x^* A x / x^* x : x \in \mathbb{C}^n \text{ and } x \ne 0\}$$

Thus, $F(\cdot)$ is a normalized version of $F'(\cdot)$.

6.   Determine $F\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $F\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $F\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7.   If $A \in M_n$ and $\alpha \in F(A)$, show that there is a unitary matrix $U \in M_n$ such that α is the 1,1 entry of $U^* A U$.

8.   Determine as many different possible types of sets as you can that can be an $F'(A)$.

9.   Show that $F(A^*) = \overline{F(A)}$ and $F'(A^*) = \overline{F'(A)}$ for all $A \in M_n$.

10.   Show that all of the main diagonal entries and eigenvalues of a given $A \in M_n$ are in its field of values $F(A)$.

1.2   Basic properties of the field of values

As a function from M_n into subsets of $\mathbb{C}$, the field of values $F(\cdot)$ has many useful functional properties, most of which are easily established. We catalog many of these properties here for reference and later use. The important property of convexity is left for discussion in the next section.

The sum or product of two subsets of $\mathbb{C}$, or of a subset of $\mathbb{C}$ and a scalar, has the usual algebraic meaning. For example, if $S, T \subset \mathbb{C}$, then $S + T \equiv \{s + t : s \in S, t \in T\}$.

1.2.1   Property: *Compactness*. For all $A \in M_n$,

$F(A)$ is a compact subset of $\mathbb{C}$

Proof: The set $F(A)$ is the range of the continuous function $x \to x^* Ax$ over the domain $\{x : x \in \mathbb{C}^n, x^*x = 1\}$, the surface of the Euclidean unit ball, which is a compact set. Since the continuous image of a compact set is compact, it follows that $F(A)$ is compact.   ☐

1.2.2   Property: *Convexity*. For all $A \in M_n$,

$F(A)$ is a convex subset of $\mathbb{C}$

The next section of this chapter is reserved for a proof of this fundamental fact, known as the *Toeplitz–Hausdorff theorem*. At this point, it is clear that $F(A)$ must be a *connected* set since it is the continuous image of a connected set.

Exercise. If A is a diagonal matrix, show that $F(A)$ is the convex hull of the diagonal entries (the eigenvalues) of A.

The field of values of a matrix is changed in a simple way by adding a scalar multiple of the identity to it or by multiplying it by a scalar.