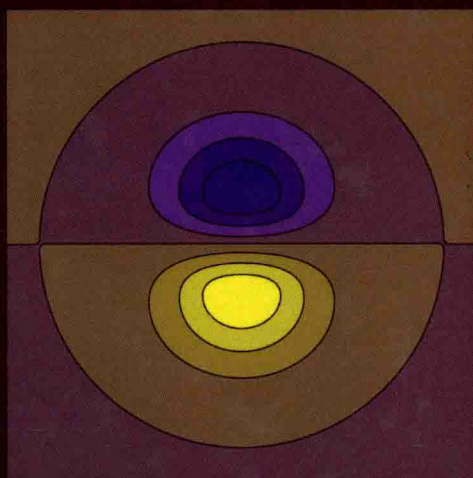
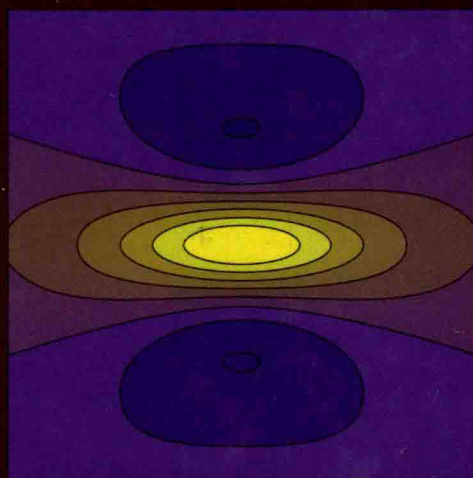


STATIC GREEN'S FUNCTIONS in ANISOTROPIC MEDIA

Ernian Pan and Weiqiu Chen



Static Green's Functions in Anisotropic Media

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STATIC GREEN'S FUNCTIONS IN ANISOTROPIC MEDIA

This book provides the basic theory on static Green's functions in general anisotropic magneto-electroelastic media and their detailed derivations based on the complex variable method, potential method, and integral transforms. Green's functions corresponding to the reduced cases are also presented, including those in anisotropic and transversely isotropic piezoelectric and piezomagnetic media and those in purely anisotropic elastic, transversely isotropic elastic, and isotropic elastic media. Addressed problem domains are three-dimensional (two-dimensional) infinite, half, and bimaterial spaces (planes). Although the emphasis is on the Green's functions related to the line and point force, those corresponding to the important line and point dislocation are also provided and discussed. This book provides a comprehensive derivation and collection of the Green's functions in the concerned media, and as such, it should be a good reference book for researchers and engineers and a textbook and reference book for both undergraduate and graduate students in engineering and applied mathematics.

Ernian Pan is a professor of civil engineering at the University of Akron and a Fellow of both ASME and ASCE. He received his BS from Lanzhou University, his MS from Peking University, and his PhD from the University of Colorado at Boulder. His research interests are in computational mechanics with applications to anisotropic magneto-electroelastic solids and nanostructures. As a well-recognized expert in anisotropic and multilayered Green's functions, he has pioneered various benchmark solutions for multiphase and multilayered composites with functionally graded materials and played various active roles in, and contributed to, Green's function research and education. He has published more than 250 journal articles.

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In the other class of methods the quantities to be determined are expressed by definite integrals, the elements of the integrals representing the effects of *singularities* distributed over the surface or through the volume. This class of solutions constitutes an extension of the methods introduced by Green in the Theory of the Potential.

—A. E. H. Love, 1944

Preface

As one of the most powerful computational methods, boundary integral equation method (with its discretized version being called boundary element method, or BEM), has been very successfully applied to various practical engineering problems. The BEM has also become a regular senior-level graduate and postgraduate course in various engineering disciplines. Because Green's functions are the key elements in the BEM approach, their derivations and behaviors are important to researchers as well as to students in almost all branches of science and engineering. With advanced materials/composites of general anisotropy being created and fabricated, and novel devices of multiphase coupling being designed, new Green's functions in anisotropic and multiphase materials are in need.

This book is intended to provide the basic theory on static Green's functions in general anisotropic magneto-electroelastic media and their detailed derivations based on the complex variable method, potential method, and integral transforms. Green's functions corresponding to the reduced (simple) cases are also presented including those in anisotropic and transversely isotropic piezoelectric and piezomagnetic media, and those in purely anisotropic elastic, transversely isotropic elastic and isotropic elastic media. Addressed problem domains are three-dimensional (two-dimensional) infinite, half, and bimaterial spaces (planes). While the emphasis is on the Green's functions related to the line and point forces (the first-order source), those corresponding to the important (line and point) dislocation source are also provided and discussed when convenient. It is the authors' intention that this book provides a relatively comprehensive derivation and collection of the Green's functions in the concerned media, and as such, it should be a good reference book in the hands of researchers and engineers, and a textbook and reference book for both undergraduate and graduate students in engineering and applied mathematics.

The book is divided into nine chapters. Chapter 1 is a brief introduction to the Green's function method and related theorems. Chapter 2 presents the governing equations, including the force and charge balance equations, generalized constitutive relations, and the gradient relations between the extended displacements and strains. While in Chapter 3 we derive the two-dimensional Green's functions in elastic isotropic full and bimaterial planes, the Green's functions in corresponding anisotropic magneto-electroelastic full and bimaterial planes are presented in Chapter 4. Chapter 5 includes the three-dimensional Green's functions in elastic

isotropic full and bimaterial spaces. While Chapter 6 derives the three-dimensional Green's functions in a transversely isotropic magneto-electroelastic full-space, the three-dimensional Green's functions in a transversely isotropic magneto-electroelastic bimaterial space are derived in Chapter 7. Chapter 8 presents the three-dimensional Green's functions in the corresponding anisotropic magneto-electroelastic full-space and Chapter 9 those in the corresponding anisotropic magneto-electroelastic bimaterial space. Direct and indirect applications of the Green's functions to various science and engineering fields are illustrated.

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The first author would like to dedicate this book to his parents. Although they never had the chance to enter school, they have provided the first author with the best possible learning opportunity!

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1 Introduction

1.0 Introduction

Green's function is named after George Green for his fundamental contributions to potential theory, reciprocal relations, singular functions, and representative theorem. Green's functions can be extremely powerful in solving various differential equations and are also the essential components in the boundary integral equation method. As singular solutions to certain differential equations in their most generalized mathematical form, such solutions can find applications in nearly every field of science and engineering. They have been and will be continuously utilized in earth science/geophysics; civil, mechanical, and aerospace engineering; physics and material science; nanoscience/nanotechnology; biotechnology; information technology, and so forth. In this chapter, we define the Green's function and introduce its basic features, along with derivations of some of the common Green's functions in potential problems.

1.1 Definition of Green's Function

Green was a mathematician and physicist of United Kingdom (Cannell 2001; Challis and Sheard 2003). He not only developed this powerful tool for solving linear differential equations, but also contributed to various problems in elasticity. For instance, he offered a derivation of the governing equations of elasticity without using any hypothesis on the behavior of the molecular structure of the solids, and was able to show further that twenty-one elastic constants are required in general to account for the general anisotropy of elastic property (Timoshenko 1953). He further explained how symmetry can reduce the independent number of these constants.

Green's function is also called singular function, which is the fundamental solution of a (partial or ordinary) differential equation (or system of equations) in the problem domain (usually of infinite size) where the inhomogeneous term in the equation is replaced by the Dirac delta function.

As an example, let us consider the following differential equation in a two-dimensional (2D) or three-dimensional (3D) infinite and homogeneous domain,

$$Lu(\mathbf{r}) = f(\mathbf{r}) \quad (1.1)$$